Characterization of Typical Equality Conditions in Operator Algebra Based on Weighted Average Transformations

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Innovation Declaration

This contest team declares that the submitted paper is the result of research work conducted under the guidance of the advisor. To the best of our knowledge, except for the content specifically noted in the text and listed in the acknowledgments, the paper does not contain research results that have been published or written by others. If there are any inaccuracies, I am willing to assume all related responsibilities.

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Contents

Abstract

References [1] and [2] have recently defined the weighted numerical radius of operators in different ways. This paper studies the differences and unifications of these two definitions, as well as their connection to the classical numerical radius. Regarding the numerical radius in reference [2],

$$\omega_v(A) = \sup_{\theta \in \mathbb{R}} ||ve^{i\theta}A + (1-v)e^{-i\theta}A^*||$$

where $v \in [0,1]$ and A is any bounded linear operator on a Hilbert space $\mathcal{H}$. Inspired by the definition of weighted operator norms in [1], and based on [2], we provide a new definition of a weighted operator norm. We call

$$M_v(A) = vA + (1-v)A^*$$

the **weighted average transformation** of operator A, and call

$$||A||_v \triangleq ||M_v(A)||$$

the **weighted operator norm** of operator A. We utilize the convexity of the weighted numerical radius combined with the Hadamard inequality to develop estimates for the numerical radius, particularly focusing on inequalities for the weighted numerical radius. We establish necessary and sufficient conditions for some boundary equalities such as

$$||A+B||_v{}^2 = ||A||_v^2 + ||B||_v^2$$

and

$$\omega_v(A^2+B^2) = 4\alpha \max\{\omega_v^2(A), \omega_v^2(B)\}$$

In particular, we use Example 3 to illustrate that the inequality $\omega_v(A+B) \leq \omega_v(A) + \omega_v(B)$ strengthens the results of Carmichael and Mason regarding polynomial root estimation.

Keywords: Weighted numerical radius, Weighted average transformation, Weighted operator norm, Polynomial roots.

1 Basic Symbols and Terminology

Definition 1.1 (Inner Product Space, [?, Definition 1]). Let $\mathcal{H}$ be a vector space over the field of complex numbers $\mathbb{C}$. A mapping

$$\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$$

is called an **inner product** on $\mathcal{H}$ if it satisfies:

1. **Linearity:** For any fixed $y \in \mathcal{H}$, the mapping $x \mapsto \langle x, y \rangle$ from $\mathcal{H}$ to $\mathbb{C}$ is linear.
2. **Symmetry:** For any $x, y \in \mathcal{H}$, $\langle x, y \rangle = \overline{\langle y, x \rangle}$.
3. **Non-negativity:** For any $x \in \mathcal{H}$, $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ if and only if $x = 0$.

A vector space endowed with an inner product is called an **inner product space**.

Definition 1.2 (Normed Space, Banach Space, [?, Definition 2]). Let X be a vector space over $\mathbb{C}$. A functional $|| \cdot || : X \to \mathbb{R}_+$ (where $\mathbb{R}_+ \geq 0$) is called a **norm** if it satisfies:

1. $||x|| = 0 \Leftrightarrow x = 0$
2. **Triangle inequality:** $||x + y|| \leq ||x|| + ||y||$
3. **Homogeneity:** $||\alpha x|| = |\alpha| ||x||$ for $\alpha \in \mathbb{C}, x \in X$

Then X is called a **normed space**.

A sequence $\{x_n\}$ in X is called a **Cauchy sequence** if for any $\epsilon > 0$, there exists a natural number N such that for $m, n \geq N$, $||x_n - x_m|| \leq \epsilon$. If every Cauchy sequence converges, the normed space X is called **complete**. A complete normed space is called a **Banach space**.

Definition 1.3 (Hilbert Space, [?, Definition 3]). Let $\mathcal{H}$ be an inner product space. Let $||x|| = \langle x, x \rangle^{\frac{1}{2}}$ for $x \in \mathcal{H}$. This is called the **norm derived from the inner product**, and $\mathcal{H}$ thus becomes a normed space. If this normed space is also complete, then $\mathcal{H}$ is called a **Hilbert space**.

Example 1.1 ([?, Example 1]). If $\mathcal{H} = \mathbb{C}^n$, and for $\forall x = (x_i), y = (y_i) \in \mathbb{C}^n$, the inner product is defined as $\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i}$, then $\mathbb{C}^n$ is a Hilbert space. Furthermore, $\mathcal{B}(\mathcal{H}) = \mathcal{M}_n(\mathbb{C})$, i.e., the set of all n-order complex matrices on $\mathbb{C}$.

Definition 1.4 (Linear Operator, [?, Definition 4]). Let $\mathcal{H}$ be a Hilbert space. A mapping $A : \mathcal{H} \to \mathcal{H}$ is called a **linear mapping** (or **linear operator**) if for any $\lambda \in \mathbb{C}$ and any $x, y \in \mathcal{H}$,

$$A(\lambda x + y) = \lambda Ax + Ay$$

A is called **bounded** if there exists $C \geq 0$ such that for any $x \in \mathcal{H}$,

$$||Ax|| \leq C||x||$$

We define

$$||A|| = \sup_{x \in \mathcal{H}, x \neq 0} \frac{||Ax||}{||x||}$$

and call $||A||$ the **norm** of A. The set of all bounded linear mappings (or bounded linear operators) from $\mathcal{H}$ to $\mathcal{H}$ is denoted $\mathcal{B}(\mathcal{H})$.

Definition 1.5 (Banach Algebra, [**?**, Definition 5])**.** Let $\mathcal{A}$ be a Banach space over $\mathbb{C}$, which is also an algebra, such that its multiplication satisfies

$$||xy|| \leq ||x||||y||$$

Then $\mathcal{A}$ is called a **Banach algebra** over $\mathbb{C}$.

Definition 1.6 ([**?**, Definition 6])**.** Let $\mathcal{A}$ be a Banach algebra. $x \in \mathcal{A}$ is called **invertible** if there exists $y \in \mathcal{A}$ such that $xy = yx = 1$.

Theorem 1.1 ([**?**, Theorem 1])**.** *$\mathcal{B}(\mathcal{H})$ is a Banach algebra.*

Definition 1.7 (Spectrum, [**?**, Definition 7])**.** If $A \in \mathcal{B}(\mathcal{H})$, let the set

$$\sigma(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not invertible}\}$$

Then $\sigma(A)$ is called the **spectrum** of A, and $r(A) = \sup_{\lambda \in \sigma(A)} |\lambda|$ is called the **spectral radius** of A.

Theorem 1.2 ([**?**, Theorem 2])**.** *If $A \in \mathcal{B}(\mathcal{H})$, then there exists a unique $A^* \in \mathcal{B}(\mathcal{H})$ such that*

$$\langle Ax, y \rangle = \langle x, A^* y \rangle, \quad \forall x, y \in \mathcal{H}$$

*A^* is called the **adjoint operator** of A.*

Definition 1.8 ([**?**, Definition 8])**.** If $A \in \mathcal{B}(\mathcal{H})$ and $A = A^*$, then A is called a **self-adjoint operator**. If $\sigma(A) \subseteq \mathbb{R}_+$, then A is called a **positive operator**.

Definition 1.9 ([**?**, Definition 9])**.** If $A \in \mathcal{B}(\mathcal{H})$, let the set

$$W(A) = \{\langle Ax, x \rangle : x \in \mathcal{H}, ||x|| = 1\}$$

Then $W(A)$ is called the **numerical range** of A, and

$$\omega(A) = \sup_{\lambda \in W(A)} |\lambda|$$

is called the **numerical radius** of A.

Theorem 1.3 ([**?**, Theorem 10])**.** *If $A \in \mathcal{B}(\mathcal{H})$, then A can be decomposed into a linear combination of two self-adjoint operators, i.e.,*

$$A = \mathcal{R}(A) + i\mathcal{I}(A)$$

where

$$\mathcal{R}(A) = \frac{A + A^*}{2}, \quad \mathcal{I}(A) = \frac{A - A^*}{2i}$$

*are respectively called the **real part** and **imaginary part** of A.*

2 Introduction

According to Theorem 10 from the "Basic Symbols and Terminology" section, for any $A \in \mathcal{B}(\mathcal{H})$, we have

$$A = \mathcal{R}(A) + i\mathcal{I}(A)$$

where $\mathcal{R}(A) = \frac{A+A^*}{2}$ and $\mathcal{I}(A) = \frac{A-A^*}{2i}$ are the real and imaginary parts of A, respectively. In particular, when $\mathcal{H}$ is the 1-dimensional complex Hilbert space $\mathbb{C}$, we have $\mathcal{B}(\mathcal{H}) = \mathbb{C}$. Then for any $z = a + ib \in \mathcal{B}(\mathcal{H})$, we have $z^* = \overline{z} = a - ib$, thus

$$\mathcal{R}(z) = \frac{(a+ib)+(a-ib)}{2} = a, \quad \text{and} \quad \mathcal{I}(z) = \frac{(a+ib)-(a-ib)}{2i} = b$$

From this fact, the definitions of the real and imaginary parts for a general $A \in \mathcal{B}(\mathcal{H})$ are natural.

Recently, Sheikhhosseini et al. [2] revisited the real and imaginary parts of A from the perspective of weighted real numbers. Specifically, for $A \in \mathcal{B}(\mathcal{H})$ and $v \in [0, 1]$, they defined:

$$\mathcal{R}_v(A) = vA + (1-v)A^* \tag{1}$$

$$\mathcal{I}_v(A) = \frac{vA - (1-v)A^*}{i} \tag{1}$$

These are referred to as the **weighted real part** and **weighted imaginary part** of the operator A. When $v = \frac{1}{2}$, we have $\mathcal{R}_v(A) = \mathcal{R}(A)$ and $\mathcal{I}_v(A) = \mathcal{I}(A)$. Furthermore, reference [2] defined the **weighted numerical radius** as:

$$\omega_v(A) = \sup_{\theta \in \mathbb{R}} ||\mathcal{R}_v(e^{i\theta}A)|| \tag{2}$$

This satisfies $\omega_{\frac{1}{2}}(A) = \omega(A)$ and $\omega_0(A) = \omega_1(A) = ||A|| = ||A^*||$.

On the other hand, by Definition 9 and Theorem 10, we have

$$\omega(A) = \sup_{x \in \mathcal{H}, ||x||=1} |\langle (\mathcal{R}(A) + i\mathcal{I}(A))x, x\rangle|$$

Also, from Theorem 1.6 and Theorem 10,

$$||A|| = \sup_{x,y \in \mathcal{H}, ||x||=||y||=1} |\langle Ax, y\rangle| = \sup_{x,y \in \mathcal{H}, ||x||=||y||=1} |\langle (\mathcal{R}(A) + i\mathcal{I}(A))x, y\rangle|$$

Influenced by Conde et al. [1], and combining the two equations above, another form of the weighted imaginary part was defined:

$$\mathcal{I}_v(A) = \frac{(1-v)A - vA^*}{i} \tag{3}$$

This leads to another form of weighted numerical radius $\omega_v(A)$ and weighted operator norm $||A||_v$:

$$\omega_v(A) = \sup_{x \in \mathcal{H}, ||x||=1} |\langle (\mathcal{R}_v(A) + i\mathcal{I}_v(A))x, x\rangle| = \omega((1-2v)A^* + A)$$

$$||A||_v = \sup_{x,y \in \mathcal{H}, ||x||=||y||=1} |\langle (\mathcal{R}_v(A) + i\mathcal{I}_v(A))x, y\rangle| = ||(1-2v)A^* + A||$$

From this, a natural question arises: Based on the weighted numerical radius defined in the sense of Sheikhhosseini et al. [2], what is the natural form of the weighted operator norm?

We expect the defined weighted operator norm to be a norm, while Proposition 1 indicates that the weighted operator norm in the sense of Conde et al. [1] is not necessarily a norm. Therefore, we examine the weighted numerical radius and weighted operator norm in the sense of Conde et al., which are characterized by a certain neatness. We hope that the weighted operator norm and weighted numerical radius defined in the sense of Sheikhhosseini et al. [2] also exhibit a certain form of neatness. To this end, we directly expand from (2):

$$\omega_v(A) = \sup_{\theta\in\mathbb{R}} ||\mathcal{R}_v(e^{i\theta}A)|| = \sup_{\theta\in\mathbb{R}} ||ve^{i\theta}A + (1-v)(e^{i\theta}A)^*||$$

We define the weighted operator norm in the sense of Sheikhhosseini et al.:

Definition 2.1 (Weighted Operator Norm)**.** If $A \in \mathcal{B}(\mathcal{H})$ and $v \in [0,1]$, then

$$||A||_v = ||vA + (1-v)A^*||$$

is called the **weighted operator norm** of the operator A. Clearly, $||A||_{\frac{1}{2}} \le ||A||_0 = ||A||_1 = ||A|| = ||A^*||$.

Definition 2.2 (Weighted Average Transformation)**.** If $A \in \mathcal{B}(\mathcal{H})$ and $v \in [0,1]$, we call

$$M_v(A) = vA + (1-v)A^*$$

the **weighted average transformation** of the operator A. In particular, $M_0(A) = A^*$, $M_1(A) = A$, and $M_{\frac{1}{2}}(A) = \mathcal{R}(A)$.

This paper starts with the basic properties of the weighted average transformation, relying on the Cauchy inequality, to obtain the basic estimate of the weighted operator norm $||A||_v$, namely **Theorem 1**:

$$||A||_v \le \frac{||v|A| + (1-v)|A^*||| + ||v|A^*| + (1-v)|A|||}{2} \le ||A||$$

The natural question is the necessary and sufficient condition for equality in the above inequality. For this, we establish a more general characterization of equality, namely the equality characterized by **Theorem 2**:

$$||AB||_v = ||A||||B||$$

The weighted operator norm inequality for the sum of two operators $A, B \in \mathcal{B}(\mathcal{H})$ is established in **Theorem 3**:

$$||A+B||_v \le (||M_v(A^*)M_v(A) + M_v(B^*)M_v(B)|| + 2\omega(M_v(B^*)M_v(A)))^{\frac{1}{2}} \le ||A||_v + ||B||_v$$

The necessary and sufficient condition for equality $||A+B||_v = ||A||_v + ||B||_v$ is established in **Theorem 4**. As a corollary, when $M_v(B^*)M_v(A)$ is a positive operator, there is a more concise necessary and sufficient condition for equality:

$$||M_v(B^*)M_v(A)|| = ||B||_v||A||_v$$

Theorem 5 then provides a characterization of the boundary equality $\omega(M_v(B^*)M_v(A)) = \max\{||A||_v^2, ||B||_v^2\}$, which is equivalent to the boundary equality $||A+B||_v = 2\max\{||A||_v, ||B||_v\}$. **Theorem 6** establishes the Pythagorean theorem for the weighted operator norm, namely

$$||A + B||_v^2 = ||A||_v^2 + ||B||_v^2$$

Influenced by these established operator norm equalities, we also consider similar problems for the weighted numerical radius, namely characterizing

$$\omega_v(A + B) = \omega_v(A) + \omega_v(B)$$

and its necessary and sufficient conditions. **Theorem 9** then provides the boundary equality for the weighted numerical radius of an operator product

$$\omega_v(AB) = max\{||A||^2, ||B||^2\}$$

and its necessary and sufficient conditions. Finally, we provide the necessary and sufficient conditions for the boundary equality

$$\omega_v(A^2 + B^2) = 4\alpha \max\{\omega_v^2(A), \omega_v^2(B)\}$$

On the other hand, using the basic relationship between the weighted numerical radius $\omega_v(A)$ and the weighted operator norm $||A||_v$

$$\omega_v(A) = \sup_{\theta\in\mathbb{R}} ||e^{i\theta} A||_v$$

we start with the convexity of the function $f(v) = ||A||_v$ and point out that the difference of convex functions induced by the weighted numerical radius is not necessarily a convex function, as seen in Example 2. Subsequently, using the $Hadamard - Hammer - Bullen$ inequality, we provide an estimate for the weighted numerical radius. These results, while strengthening the estimates of the weighted numerical radius by Sheikhhosseini [2] and others, also, to some extent, enhance the estimates of polynomial roots. Specifically, given an n-degree polynomial

$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

where $a_0 \neq 0$ and $a_1, \ldots, a_{n-1}$ are complex numbers, the Frobenius companion matrix $C(p)$ of $p(z)$ is

$$C(p) = \begin{pmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_0 \\ 1 & 0 & \cdots & 0 \\ \vdots & 1 & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}$$

It is well-known that the characteristic polynomial of $C(p)$ is $p(z)$, i.e., the eigenvalues of $C(p)$ are precisely the roots of $p(z)$. Thus, we have

$$\{|z| : p(z) = 0\} = \{|\lambda| : |\lambda - C(p)| = 0\} \leq r(C(p))$$

And from Theorem 2.0 in the Preliminary Knowledge, we have

$$\{|z| : p(z) = 0\} \leq r(C(p)) \leq \omega(C(p))$$

We utilize the estimation result for $\omega_v(A + B)$ from Corollary 6, setting $v = \frac{1}{2}$, to provide an inequality for $\omega(C(p)) = \omega_{\frac{1}{2}}(C(p))$. Example 3 uses this idea to give an estimate for the roots of the polynomial $p(z) = z^3 + z + 1$, which, to some extent, improves upon the estimates of polynomial roots by Carmichael and Mason [12].

3 Preliminary Knowledge

Theorem 3.1 ([?, Theorem 1.1]). *If $A \in \mathcal{B}(\mathcal{H})$, then*

$$||A|| = \sup_{x \in \mathcal{H}, ||x|| \leq 1} ||Ax|| = \sup_{x \in \mathcal{H}, ||x|| = 1} ||Ax||$$

Furthermore, for any $x \in \mathcal{H}$, $||Ax|| \leq ||A|| \cdot ||x||$.

Theorem 3.2 ([?, Theorem 1.2]). *If $A, B \in \mathcal{B}(\mathcal{H})$, then the composite mapping $A \circ B \triangleq AB \in \mathcal{B}(\mathcal{H})$, and*

$$||AB|| \leq ||A||||B||$$

Definition 3.1 ([?, Definition 1.1]). If $A \in \mathcal{B}(\mathcal{H})$, then $|A| = (A^*A)^{\frac{1}{2}}$ is called the **arithmetic square root** of A^*A.

Remark 1. Reference [8, Chapter VIII, Proposition 3.5] points out that $|A| = |A|^*$.

Theorem 3.3 ([?, Theorem 1.3]). *If $A, B \in \mathcal{B}(\mathcal{H})$, then*

1. $(A^*)^* = A$

2. $||A|| = ||A^*|| = ||AA^*||^{\frac{1}{2}} = ||A^*A||^{\frac{1}{2}} = |||A||| = |||A^*|||$

3. $(AB)^* = B^*A^*$.

Proof. We will only prove $|||A^*A||^{\frac{1}{2}} = ||A|||$; the other equalities are well-known. In fact, using $||A|| = ||A^*A||^{\frac{1}{2}}$ and $|A| = |A|^*$ leads to $|||A||| = |||A|^*||^{\frac{1}{2}} = ||A^*A||^{\frac{1}{2}}$. □

Theorem 3.4 ([?, Theorem 1.4]). *If $A \in \mathcal{B}(\mathcal{H})$, then A is a positive operator if and only if for any $x \in \mathcal{H}$, $\langle Ax, x \rangle \geq 0$.*

4 Introduction

According to Theorem 10 from the "Basic Symbols and Terminology" section, for any $A \in \mathcal{B}(\mathcal{H})$, we have

$$A = \mathcal{R}(A) + i\mathcal{I}(A)$$

where $\mathcal{R}(A) = \frac{A+A^*}{2}$ and $\mathcal{I}(A) = \frac{A-A^*}{2i}$ are the real and imaginary parts of A, respectively. In particular, when $\mathcal{H}$ is the 1-dimensional complex Hilbert space $\mathbb{C}$, we have $\mathcal{B}(\mathcal{H}) = \mathbb{C}$. Then for any $z = a + ib \in \mathcal{B}(\mathcal{H})$, we have $z^* = \overline{z} = a - ib$, thus

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From this fact, the definitions of the real and imaginary parts for a general $A \in \mathcal{B}(\mathcal{H})$ are natural.

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$$\mathcal{R}_v(A) = vA + (1-v)A^* \tag{2}$$

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These are referred to as the **weighted real part** and **weighted imaginary part** of the operator A. When $v = \frac{1}{2}$, we have $\mathcal{R}_v(A) = \mathcal{R}(A)$ and $\mathcal{I}_v(A) = \mathcal{I}(A)$. Furthermore, reference [2] defined the **weighted numerical radius** as:

$$\omega_v(A) = \sup_{\theta \in \mathbb{R}} ||\mathcal{R}_v(e^{i\theta}A)|| \tag{2}$$

This satisfies $\omega_{\frac{1}{2}}(A) = \omega(A)$ and $\omega_0(A) = \omega_1(A) = ||A|| = ||A^*||$.

On the other hand, by Definition 9 and Theorem 10, we have

$$\omega(A) = \sup_{x \in \mathcal{H}, ||x||=1} |\langle (\mathcal{R}(A) + i\mathcal{I}(A))x, x\rangle|$$

Also, from Theorem 1.6 and Theorem 10,

$$||A|| = \sup_{x,y \in \mathcal{H}, ||x||=||y||=1} |\langle Ax, y\rangle| = \sup_{x,y \in \mathcal{H}, ||x||=||y||=1} |\langle (\mathcal{R}(A) + i\mathcal{I}(A))x, y\rangle|$$

Influenced by Conde et al. [1], and combining the two equations above, another form of the weighted imaginary part was defined:

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This leads to another form of weighted numerical radius $\omega_v(A)$ and weighted operator norm $||A||_v$:

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From this, a natural question arises: Based on the weighted numerical radius defined in the sense of Sheikhhosseini et al. [2], what is the natural form of the weighted operator norm?

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We define the weighted operator norm in the sense of Sheikhhosseini et al.:

Definition 4.1 (Weighted Operator Norm)**.** If $A \in \mathcal{B}(\mathcal{H})$ and $v \in [0,1]$, then

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Definition 4.2 (Weighted Average Transformation)**.** If $A \in \mathcal{B}(\mathcal{H})$ and $v \in [0,1]$, we call

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The necessary and sufficient condition for equality $||A+B||_v = ||A||_v + ||B||_v$ is established in **Theorem 4**. As a corollary, when $M_v(B^*)M_v(A)$ is a positive operator, there is a more concise necessary and sufficient condition for equality:

$$||M_v(B^*)M_v(A)|| = ||B||_v||A||_v$$

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On the other hand, using the basic relationship between the weighted numerical radius $\omega_v(A)$ and the weighted operator norm $||A||_v$

$$\omega_v(A) = \sup_{\theta\in\mathbb{R}} ||e^{i\theta}A||_v$$

we start with the convexity of the function $f(v) = ||A||_v$ and point out that the difference of convex functions induced by the weighted numerical radius is not necessarily a convex function, as seen in Example 2. Subsequently, using the $Hadamard - Hammer - Bullen$ inequality, we provide an estimate for the weighted numerical radius. These results, while strengthening the estimates of the weighted numerical radius by Sheikhhosseini [2] and others, also, to some extent, enhance the estimates of polynomial roots. Specifically, given an n-degree polynomial

$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

where $a_0 \neq 0$ and $a_1, \ldots, a_{n-1}$ are complex numbers, the Frobenius companion matrix $C(p)$ of $p(z)$ is

$$C(p) = \begin{pmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_0 \\ 1 & 0 & \cdots & 0 \\ \vdots & 1 & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}$$

It is well-known that the characteristic polynomial of $C(p)$ is $p(z)$, i.e., the eigenvalues of $C(p)$ are precisely the roots of $p(z)$. Thus, we have

$$\{|z| : p(z) = 0\} = \{|\lambda| : |\lambda - C(p)| = 0\} \leq r(C(p))$$

And from Theorem 2.0 in the Preliminary Knowledge, we have

$$\{|z| : p(z) = 0\} \leq r(C(p)) \leq \omega(C(p))$$

We utilize the estimation result for $\omega_v(A + B)$ from Corollary 6, setting $v = \frac{1}{2}$, to provide an inequality for $\omega(C(p)) = \omega_{\frac{1}{2}}(C(p))$. Example 3 uses this idea to give an estimate for the roots of the polynomial $p(z) = z^3 + z + 1$, which, to some extent, improves upon the estimates of polynomial roots by Carmichael and Mason [12].

5 Preliminary Knowledge

Theorem 5.1 ([?, Theorem 1.1]). *If $A \in \mathcal{B}(\mathcal{H})$, then*

$$||A|| = \sup_{x \in \mathcal{H}, ||x|| \leq 1} ||Ax|| = \sup_{x \in \mathcal{H}, ||x|| = 1} ||Ax||$$

Furthermore, for any $x \in \mathcal{H}$, $||Ax|| \leq ||A|| \cdot ||x||$.

Theorem 5.2 ([?, Theorem 1.2]). *If $A, B \in \mathcal{B}(\mathcal{H})$, then the composite mapping $A \circ B \triangleq AB \in \mathcal{B}(\mathcal{H})$, and*

$$||AB|| \leq ||A|| ||B||$$

Definition 5.1 ([?, Definition 1.1]). If $A \in \mathcal{B}(\mathcal{H})$, then $|A| = (A^*A)^{\frac{1}{2}}$ is called the **arithmetic square root** of A^*A.

Remark 2. Reference [8, Chapter VIII, Proposition 3.5] points out that $|A| = |A|^*$.

Theorem 5.3 ([?, Theorem 1.3]). *If $A, B \in \mathcal{B}(\mathcal{H})$, then*

1. $(A^*)^* = A$
2. $||A|| = ||A^*|| = ||AA^*||^{\frac{1}{2}} = ||A^*A||^{\frac{1}{2}} = |||A||| = |||A^*|||$
3. $(AB)^* = B^*A^*$.

Proof. We will only prove $|||A^*A||^{\frac{1}{2}} = ||A|||$; the other equalities are well-known. In fact, using $||A|| = ||A^*A||^{\frac{1}{2}}$ and $|A| = |A|^*$ leads to $|||A||| = |||A|^*||^{\frac{1}{2}} = ||A^*A||^{\frac{1}{2}}$. □

Theorem 5.4 ([?, Theorem 1.4]). *If $A \in \mathcal{B}(\mathcal{H})$, then A is a positive operator if and only if for any $x \in \mathcal{H}$, $\langle Ax, x \rangle \geq 0$.*

Theorem 5.5 (Cauchy-Schwarz Inequality, [?, Theorem 1.5]). *Let $\mathcal{H}$ be an inner product space. Then for any $x, y \in \mathcal{H}$, we have*

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle$$

Furthermore, the equality holds if and only if x and y are proportional.

Theorem 5.6 ([**?**, Theorem 1.6]). *If $A \in \mathcal{B}(\mathcal{H})$, then*

$$||A|| = \sup_{x,y \in \mathcal{H}, ||x||=||y||=1} |\langle Ax, y\rangle|$$

Proof. For any $x, y \in \mathcal{H}$ with $||x|| = ||y|| = 1$, the Cauchy-Schwarz inequality gives

$$|\langle Ax, y\rangle| \leq ||Ax|| \cdot ||y|| = ||Ax|| \leq ||A||$$

On the other hand, taking $y = \frac{Ax}{||Ax||}$ (assuming $Ax \neq 0$), we have $|\langle Ax, y\rangle| = ||Ax||$. Thus,

$$||Ax|| \leq \sup_{x,y \in \mathcal{H}, ||x||=||y||=1} |\langle Ax, y\rangle| \leq ||A||$$

Taking the supremum of the left side with respect to $||x|| = 1$, by Theorem 1.1 we have $||A|| \leq \sup |\langle Ax, y\rangle|$, which proves the equality. □

Theorem 5.7 ([**?**, Theorem 1.7]). *If $A \in \mathcal{B}(\mathcal{H})$ is a normal operator, i.e., $AA^* = A^*A$, then*

$$r(A) = \omega(A) = ||A||$$

Theorem 5.8 ([**?**, Theorem 1.8]). *If $A \in \mathcal{B}(\mathcal{H})$, then $\omega(A) = \sup_{\theta \in \mathbb{R}} ||\mathcal{R}(e^{i\theta}A)||$.*

Theorem 5.9 (Mixed Cauchy-Schwarz Inequality, [**?**, Theorem 1.9]). *If $A \in \mathcal{B}(\mathcal{H})$, and $x, y \in \mathcal{H}$, then*

$$|\langle Ax, y\rangle| \leq \langle |A|x, x\rangle^{\frac{1}{2}} \langle |A^*|y, y\rangle^{\frac{1}{2}}$$

Theorem 5.10 ([**?**, Theorem 2.0]). *If $A \in \mathcal{B}(\mathcal{H})$, then $r(A) \leq \omega(A) \leq ||A||$.*

Theorem 5.11 ([**?**, Theorem 2.1]). *If $A \in \mathcal{B}(\mathcal{H})$, then $\omega(A^2) \leq \omega^2(A)$.*

Theorem 5.12 ([**?**, Theorem 2.2]). *If $A, B \in \mathcal{B}(\mathcal{H})$ and $v \in [0, 1]$, then $\omega_v(A + B) \leq \omega_v(A) + \omega_v(B)$.*

Definition 5.2 ([**?**, Definition 1.2]). Let f be a function defined on an interval I. If for any two points x_1, x_2 on I and any real number $\lambda \in (0, 1)$, it always holds that

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2)$$

then f is called a **convex function** on I.

Lemma 5.13 ([**?**, Lemma 1.1]). *A necessary and sufficient condition for f to be a convex function on I is: for any three points $x_1 < x_2 < x_3$ on I, it always holds that*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

Theorem 5.14 (Hadamard-Hammer-Bullen Inequality, [?, Theorem 2.3]). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function. Then for $x, y \in I$ and $\lambda \in [0,1]$:*

$$\begin{aligned} f\left(\frac{x+y}{2}\right) &\leq (1-\lambda) f\left(\frac{(1-\lambda)x + (1+\lambda)y}{2}\right) + \lambda f\left(\frac{(2-\lambda)x + \lambda y}{2}\right) \\ &\leq \int_0^1 f((1-t)x + ty)dt \\ &\leq \frac{1}{2}\left(f((1-\lambda)x + \lambda y) + (1-\lambda)f(y) + \lambda f(x)\right) \\ &\leq \frac{f(x)+f(y)}{2} \end{aligned}$$

Theorem 5.15 ([?, Theorem 2.4]). *If $A \in \mathcal{B}(\mathcal{H})$ and $v \in [\frac{1}{2}, 1]$, then $\omega_v(A) \leq 2v\omega(A)$.*

6 Main Results

We begin with the necessary and sufficient conditions for the two types of weighted operator norms to be norms, and then provide results regarding the weighted average transformation in the sense of Sheikhhosseini.

Proposition 6.1. *Let $v \in [0,1]$. In the sense of Conde et al. [1], the function $||\cdot||_v : \mathcal{B}(\mathcal{H}) \to \mathbb{R}_+$ defined by*

$$A \mapsto ||(1-2v)A^* + A||$$

is a norm if and only if $v = \frac{1}{2}$.

Proof. If $v = \frac{1}{2}$, then for any $A \in \mathcal{B}(\mathcal{H})$, we have

$$||A||_v = ||(1 - 2\cdot\frac{1}{2})A^* + A|| = ||A||$$

Thus, $||\cdot||_v$ is obviously a norm.

If $v \neq \frac{1}{2}$, we assert that the function $||\cdot||_v$ does not satisfy the homogeneity of norms, and thus $||\cdot||_v$ is not a norm. In fact, considering the pure imaginary unit i and the identity operator I, we have

$$||iI||_v = ||(1-2v)(iI)^* + iI|| = ||(1-2v)(-iI) + iI|| = ||(-i + 2vi + i)I|| = ||2viI|| = 2v$$

and

$$|i| \cdot ||I||_v = 1 \cdot ||(1-2v)I^* + I|| = ||(1-2v)I + I|| = ||(2-2v)I|| = |2-2v| = 2(1-v)$$

Since $v \neq \frac{1}{2}$, $2v \neq 2(1-v)$, and therefore $||iI||_v \neq |i| \cdot ||I||_v$. Thus, $||\cdot||_v$ is not a norm. □

Proposition 6.2. *Let $v \in [0,1]$. In the sense of Sheikhhosseini, the function $||\cdot||_v : \mathcal{B}(\mathcal{H}) \to \mathbb{R}_+$ defined by*

$$A \mapsto ||vA + (1-v)A^*||$$

is a norm if and only if $v = 0$ or $v = 1$.

Proof. If $v = 0$ or $v = 1$, then for any $A \in \mathcal{B}(\mathcal{H})$, we have

$$||A||_v = ||A^*|| = ||A|| \quad \text{(for v=0 or v=1)}$$

Thus, $||\cdot||_v$ is obviously a norm.

If $v \neq 0, 1$, we assert that the function $||\cdot||_v$ does not satisfy the homogeneity of norms, and thus $||\cdot||_v$ is not a norm. In fact, considering the pure imaginary unit i and the identity operator I, we have

$$||iI||_v = ||v(iI)+(1-v)(iI)^*|| = ||viI+(1-v)(-iI)|| = ||(vi-i+vi)I|| = ||(2v-1)iI|| = |2v-1|$$

and

$$|i| \cdot ||I||_v = 1 \cdot ||vI + (1-v)I|| = ||(v + 1 - v)I|| = ||I|| = 1$$

Since $v \neq 0, 1$, we have $v \neq 1/2$, so $|2v - 1| \neq 1$. Therefore, $||iI||_v \neq |i| \cdot ||I||_v$, i.e., $||\cdot||_v$ is not a norm. □

Now, we begin our main object of study—the weighted average transformation of operators, starting with some simple properties.

Proposition 6.3. *If $A \in \mathcal{B}(\mathcal{H})$, then the following conditions are equivalent:*

1. *A is a self-adjoint operator ($A = A^*$).*
2. *For any $v \in (0, 1)$, $M_v(A) = A$ holds.*
3. *There exists $v \in (0, 1)$ such that $M_v(A) = A$ holds.*
4. *There exists $v \in (0, 1)$ such that $M_v(A) = A^*$ holds.*
5. *For any $v \in (0, 1)$, $M_v(A) = A^*$ holds.*

Proof. Note that when $v \neq 1$, we have

$$M_v(A) = A \Leftrightarrow vA + (1-v)A^* = A \Leftrightarrow (1-v)A^* = (1-v)A \Leftrightarrow A = A^*$$

And when $v \neq 0$, we have

$$M_v(A) = A^* \Leftrightarrow vA + (1-v)A^* = A^* \Leftrightarrow vA = vA^* \Leftrightarrow A = A^*$$

- (1) $\Rightarrow$ (2): Because $A = A^*$, for any $v \in (0, 1)$, we have $M_v(A) = vA + (1-v)A^* = vA + (1-v)A = A$.
- (2) $\Rightarrow$ (3): This is evident.
- (3) $\Rightarrow$ (4): Because $\exists v \in (0, 1)$ such that $M_v(A) = A$, by the initial observation, $A = A^*$. Thus $M_v(A) = A = A^*$.
- (4) $\Rightarrow$ (5): Because $\exists v \in (0, 1)$ such that $M_v(A) = A^*$, by the initial observation, $A = A^*$. Therefore, for any $v \in (0, 1)$, we have $M_v(A) = vA + (1-v)A^* = vA^* + (1-v)A^* = A^*$.

- (5) $\Rightarrow$ (1): By the initial observation, this is evident.

$\square$

Remark 3. Simply, take $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_2(\mathbb{C})$. We have $M_v(A) = \begin{cases} A & v = 1 \\ A^* & v = 0 \end{cases}$, and $A \neq A^*$.

Proposition 6.4. *If $A \in \mathcal{B}(\mathcal{H})$ and $v \neq \frac{1}{2}$, then $M_v(A) = 0$ if and only if $A = 0$.*

Proof. When $v = 1$, $M_v(A) = A$, and the proposition is evidently true. When $v \neq 1$, if $M_v(A) = 0$, then

$$vA + (1 - v)A^* = 0 \tag{4}$$

By the properties of equality,

$$A^* = \frac{v}{v-1}A \tag{5}$$

Taking the * operation on both sides of (4), by Theorem 1.3(1) we have

$$vA^* + (1 - v)A = 0 \tag{6}$$

Substituting (5) into (6) gives

$$\left(\frac{v^2}{v-1} + (1 - v)\right) A = \left(\frac{v^2 - (1 - v)^2}{v - 1}\right) A = \left(\frac{2v - 1}{v - 1}\right) A = 0$$

Since $v \neq \frac{1}{2}$, $\frac{v^2}{v-1} + (1 - v) \neq 0$. Thus, from the above equation, we have $A = 0$. The converse is evidently true. $\square$

Remark 4. When $v = \frac{1}{2}$, it is easy to see that $M_v(A) = 0$ if and only if $A = -A^*$, i.e., A is an anti-self-adjoint operator.

Next, we study the norm properties of $M_v(A)$, i.e., the properties of the weighted operator norm $||A||_v = ||M_v(A)||$. The following theorem provides an estimate for $||A||_v$ and shows that $M_v(A)$ is a contractive transformation.

Theorem 6.5. *If $A, B \in \mathcal{B}(\mathcal{H})$ and $v \in [0, 1]$, then*

$$\begin{aligned} ||A + B||_v &\leq (||M_v(A^*)M_v(A) + M_v(B^*)M_v(B)|| + 2\omega(M_v(B^*)M_v(A)))^{\frac{1}{2}} \\ &\leq ||A||_v + ||B||_v \end{aligned}$$

Proof. For any $x \in \mathcal{H}$ with $||x|| = 1$, we expand the norm:

$$\begin{aligned}
&||v(A+B)x + (1-v)(A+B)^*x||^2 \\
&\quad = \langle v(A+B)x + (1-v)(A+B)^*x, v(A+B)x + (1-v)(A+B)^*x \rangle \\
&\quad = \ldots \text{(expansion of many terms)} \ldots \\
&\quad \le v^2\langle Ax, Ax\rangle + (1-v)^2\langle A^*x, A^*x\rangle + v^2\langle Bx, Bx\rangle + (1-v)^2\langle B^*x, B^*x\rangle \\
&\quad\quad + v(1-v)2\mathcal{R}(\langle A^*x, Ax\rangle + \langle B^*x, Bx\rangle) \\
&\quad\quad + 2\omega((vB^* + (1-v)B)(vA + (1-v)A^*)) \\
&\quad = \ldots \text{(regrouping terms)} \ldots \\
&\quad \le ||(vA^* + (1-v)A)(vA + (1-v)A^*) + (vB^* + (1-v)B)(vB + (1-v)B^*)|| \\
&\quad\quad + 2\omega((vB^* + (1-v)B)(vA + (1-v)A^*)) \\
&\quad = ||M_v(A^*)M_v(A) + M_v(B^*)M_v(B)|| + 2\omega(M_v(B^*)M_v(A)) \\
&\quad \le ||M_v(A^*)M_v(A)|| + ||M_v(B^*)M_v(B)|| + 2\omega(M_v(B^*)M_v(A)) \\
&\quad = ||vA + (1-v)A^*||^2 + ||vB + (1-v)B^*||^2 + 2\omega(M_v(B^*)M_v(A)) \\
&\quad \le ||A||_v^2 + ||B||_v^2 + 2||M_v(B^*)M_v(A)|| \\
&\quad \le ||A||_v^2 + ||B||_v^2 + 2||M_v(B^*)|| \cdot ||M_v(A)|| \\
&\quad = ||A||_v^2 + ||B||_v^2 + 2||B||_v||A||_v \\
&\quad = (||A||_v + ||B||_v)^2
\end{aligned}$$

Taking the supremum over x and then the square root of both sides gives the result. $\square$

Next, to provide the conditions for equality in Theorem 3, we have the following lemma.

Lemma 6.6. *Let $\{z_n = \mathcal{R}z_n + i\mathcal{I}z_n\}$ be a sequence of complex numbers, and a be a non-negative real number. If $\lim_{n\to\infty} \mathcal{R}z_n = a$ and $\lim_{n\to\infty} |z_n| = a$, then $\lim_{n\to\infty} z_n = a$.*

Proof. The case $a = 0$ is obvious, so we prove the case $a \neq 0$. Since $\lim_{n\to\infty} \mathcal{R}z_n = a$, for any $\epsilon \in (0, a)$, $\exists N_1$ such that $\forall n \ge N_1$, we have $|\mathcal{R}z_n - a| < \frac{\epsilon}{2}$, which implies $a - \frac{\epsilon}{2} < \mathcal{R}z_n < a + \frac{\epsilon}{2}$.

Also, $\lim_{n\to\infty} |z_n| = a$, so $\exists N_2 \ge N_1$ such that $\forall n \ge N_2$, we have $||z_n| - a| < \epsilon$, or $a - \epsilon < |z_n| < a + \epsilon$.

We have $|z_n|^2 = |\mathcal{R}z_n|^2 + |\mathcal{I}z_n|^2$. Thus,

$$|\mathcal{I}z_n|^2 = |z_n|^2 - |\mathcal{R}z_n|^2 < (a+\epsilon)^2 - \left(a - \frac{\epsilon}{2}\right)^2 = (a^2 + 2a\epsilon + \epsilon^2) - (a^2 - a\epsilon + \frac{\epsilon^2}{4}) = 3a\epsilon + \frac{3}{4}\epsilon^2$$

And

$$|\mathcal{I}z_n|^2 = |z_n|^2 - |\mathcal{R}z_n|^2 > (a-\epsilon)^2 - \left(a + \frac{\epsilon}{2}\right)^2 = (a^2 - 2a\epsilon + \epsilon^2) - (a^2 + a\epsilon + \frac{\epsilon^2}{4}) = -3a\epsilon + \frac{3}{4}\epsilon^2$$

Since ϵ can be arbitrarily small, this implies

$$\lim_{n\to\infty} \mathcal{I}z_n = 0 \tag{13}$$

Now, $|z_n - a| = |(\mathcal{R}z_n - a) + i\mathcal{I}z_n| \le |\mathcal{R}z_n - a| + |\mathcal{I}z_n|$. Combining this with (13) and $\lim_{n\to\infty} \mathcal{R}z_n = a$, it is easy to see that $\lim_{n\to\infty} z_n = a$. $\square$

Theorem 6.7. *If $A, B \in \mathcal{B}(\mathcal{H})$ and $v \in [0,1]$, then the following propositions are equivalent:*

- *(i)* $||A+B||_v = ||A||_v + ||B||_v$
- *(ii) There exists a sequence of unit vectors* $\{x_n\}$ *in* $\mathcal{H}$ *such that*

$$\lim_{n\to\infty} \langle M_v(A)x_n, M_v(B)x_n\rangle = ||A||_v||B||_v$$

Proof. $(i) \Rightarrow (ii)$: Assume $||A+B||_v = ||A||_v + ||B||_v$. By definition, $||A+B||_v = \sup_{||x||=1} ||M_v(A+B)x||$. Thus, there exists a unit sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} ||(v(A+B) + (1-v)(A+B)^*)x_n|| = ||A||_v + ||B||_v \tag{14}$$

On the other hand, by the triangle inequality:

$$\begin{aligned} ||M_v(A+B)x_n|| &= ||M_v(A)x_n + M_v(B)x_n|| \\ &\leq ||M_v(A)x_n|| + ||M_v(B)x_n|| \\ &\leq ||M_v(A)|| + ||M_v(B)|| = ||A||_v + ||B||_v \end{aligned}$$

Since the limit of the first term is $||A||_v + ||B||_v$, all inequalities must become equalities in the limit. This implies $\lim_{n\to\infty}(||M_v(A)x_n|| + ||M_v(B)x_n||) = ||A||_v + ||B||_v$, which in turn implies:

$$\lim_{n\to\infty} ||M_v(A)x_n|| = ||A||_v \tag{15}$$

and

$$\lim_{n\to\infty} ||M_v(B)x_n|| = ||B||_v \tag{16}$$

Furthermore, we expand the squared norm:

$$\begin{aligned} ||M_v(A+B)x_n||^2 &= ||M_v(A)x_n + M_v(B)x_n||^2 \\ &= ||M_v(A)x_n||^2 + ||M_v(B)x_n||^2 + 2\mathcal{R}\langle M_v(A)x_n, M_v(B)x_n\rangle \end{aligned}$$

Taking the limit $n \to \infty$ and using (14), (15), and (16):

$$(||A||_v + ||B||_v)^2 = ||A||_v^2 + ||B||_v^2 + 2\lim_{n\to\infty} \mathcal{R}\langle M_v(A)x_n, M_v(B)x_n\rangle$$

This implies $\lim_{n\to\infty} \mathcal{R}\langle M_v(A)x_n, M_v(B)x_n\rangle = ||A||_v||B||_v$.

Also, by Cauchy-Schwarz, $|\langle M_v(A)x_n, M_v(B)x_n\rangle| \leq ||M_v(A)x_n|| \cdot ||M_v(B)x_n||$. Taking the limit gives $\lim_{n\to\infty} |\langle M_v(A)x_n, M_v(B)x_n\rangle| \leq ||A||_v||B||_v$. Since the limit of the real part is $||A||_v||B||_v$, we must also have $\lim_{n\to\infty} |\langle M_v(A)x_n, M_v(B)x_n\rangle| = ||A||_v||B||_v$. By Lemma 2, since the limit of the real part and the limit of the modulus are equal, the limit of the complex number itself is the real number:

$$\lim_{n\to\infty} \langle M_v(A)x_n, M_v(B)x_n\rangle = ||A||_v||B||_v$$

□

Continuation of Theorem 4 Proof. (*ii*) $\Rightarrow$ (*i*): Assume there exists a unit sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} \langle M_v(A)x_n, M_v(B)x_n\rangle = ||A||_v||B||_v$$

By Lemma 1, this implies

$$\lim_{n\to\infty} \mathcal{R}\langle M_v(A)x_n, M_v(B)x_n\rangle = ||A||_v||B||_v \tag{17}$$

By Cauchy-Schwarz,

$$|\langle M_v(A)x_n, M_v(B)x_n\rangle| \leq ||M_v(A)x_n|| \cdot ||M_v(B)x_n|| \leq ||A||_v||B||_v$$

Since the limit of the inner product is $||A||_v||B||_v$, the inequalities must be equalities in the limit. This implies:

$$\lim_{n\to\infty} ||M_v(A)x_n|| = ||A||_v \tag{18}$$

and

$$\lim_{n\to\infty} ||M_v(B)x_n|| = ||B||_v \tag{19}$$

Now, using (17), (18), and (19), we consider the limit of the squared sum:

$$\begin{aligned}
(||A||_v + ||B||_v)^2 &= \lim_{n\to\infty} \left(||M_v(A)x_n||^2 + ||M_v(B)x_n||^2 + 2\mathcal{R}\langle M_v(A)x_n, M_v(B)x_n\rangle\right) \\
&= \lim_{n\to\infty} ||M_v(A)x_n + M_v(B)x_n||^2 \\
&= \lim_{n\to\infty} ||(vA + (1-v)A^*)x_n + (vB + (1-v)B^*)x_n||^2 \\
&= \lim_{n\to\infty} ||M_v(A+B)x_n||^2 \\
&\leq \sup_{||x||=1} ||M_v(A+B)x||^2 = ||A+B||_v^2
\end{aligned}$$

So, $(||A||_v + ||B||_v)^2 \leq ||A+B||_v^2$.

By the triangle inequality, we also know $||A+B||_v \leq ||A||_v + ||B||_v$. Therefore, we must have $||A+B||_v = ||A||_v + ||B||_v$. □

Remark 5. Before the next result, we emphasize that for $\forall A \in \mathcal{B}(\mathcal{H}), v \in [0,1]$, it holds that

$$(M_v(A))^* = M_v(A^*)$$

In fact, $(M_v(A))^* = (vA + (1-v)A^*)^* = vA^* + (1-v)(A^*)^* = vA^* + (1-v)A = M_v(A^*)$.

Corollary 6.8. *If $A, B \in \mathcal{B}(\mathcal{H})$, the following propositions are equivalent:*

- *(i) $||A+B|| = ||A|| + ||B||$*
- *(ii) There exists a sequence of unit vectors $\{x_n\}$ in $\mathcal{H}$ such that*

$$\lim_{n\to\infty} \langle Ax_n, Bx_n\rangle = ||A||||B||$$

Proof. Take $v = 1$ in Theorem 4. □

Theorem 6.9. *If $A, B \in \mathcal{B}(\mathcal{H})$, $v \in [0,1]$, and $M_v(B^*)M_v(A)$ is a positive operator, then the following conditions are equivalent:*

- *(i)* $||A+B||_v = 2\max\{||A||_v, ||B||_v\}$
- *(ii)* $\omega(M_v(B^*)M_v(A)) = \max\{||A||_v^2, ||B||_v^2\}$

Proof. $(i) \Rightarrow (ii)$: Given $||A+B||_v = 2\max\{||A||_v, ||B||_v\}$. We know from the triangle inequality that

$$\begin{aligned} ||A+B||_v &= ||M_v(A+B)|| = ||M_v(A) + M_v(B)|| \\ &\leq ||M_v(A)|| + ||M_v(B)|| = ||A||_v + ||B||_v \\ &\leq 2\max\{||A||_v, ||B||_v\} \end{aligned}$$

Since the first and last terms are equal, all inequalities must be equalities. This implies:

$$||A||_v = ||B||_v = \max\{||A||_v, ||B||_v\} \quad \text{and} \quad ||A+B||_v = ||A||_v + ||B||_v$$

From $||A+B||_v = ||A||_v + ||B||_v$ and Corollary 3, we have

$$||M_v(B^*)M_v(A)|| = ||B||_v||A||_v$$

Since $||A||_v = ||B||_v$, this becomes $||M_v(B^*)M_v(A)|| = \max\{||A||_v^2, ||B||_v^2\}$. Because $M_v(B^*)M_v(A)$ is positive, its numerical radius equals its norm:

$$\omega(M_v(B^*)M_v(A)) = ||M_v(B^*)M_v(A)|| = \max\{||A||_v^2, ||B||_v^2\}$$

$(ii) \Rightarrow (i)$: Assume $\omega(M_v(B^*)M_v(A)) = \max\{||A||_v^2, ||B||_v^2\}$. We have the chain of inequalities:

$$\begin{aligned} \max\{||A||_v^2, ||B||_v^2\} &= \omega(M_v(B^*)M_v(A)) \quad \text{(by assumption)} \\ &\leq ||M_v(B^*)M_v(A)|| \quad \text{(by Theorem 2.0)} \\ &\leq ||M_v(B^*)|| \cdot ||M_v(A)|| \quad \text{(by Theorem 1.2)} \\ &= ||B||_v||A||_v \quad \text{(by Remark)} \\ &\leq \frac{||A||_v^2 + ||B||_v^2}{2} \quad \text{(by AM-GM inequality)} \\ &\leq \max\{||A||_v^2, ||B||_v^2\} \end{aligned}$$

All inequalities must be equalities. This implies:

1. $||A||_v = ||B||_v = \max\{||A||_v, ||B||_v\}$ (from the AM-GM step)
2. $\omega(M_v(B^*)M_v(A)) = ||M_v(B^*)M_v(A)|| = ||B||_v||A||_v$

Since $M_v(B^*)M_v(A)$ is positive and (ii) of Corollary 3 is satisfied, we conclude that

$$||A+B||_v = ||A||_v + ||B||_v$$

Combining this with $||A||_v = ||B||_v$ gives

$$||A+B||_v = 2\max\{||A||_v, ||B||_v\}$$

□

Example 6.1. The condition "$M_v(B^*)M_v(A)$ is a positive operator" in Theorem 5 cannot be omitted. Let $v = 1$, $B = -I$, $A = I$. Then $M_v(B^*) = M_1((-I)^*) = -I$ and $M_v(A) = M_1(I) = I$. Thus, $M_v(B^*)M_v(A) = -I$. Condition (ii) would be $\omega(-I) = \max\{||I||_1^2, ||-I||_1^2\} = \max\{1, 1\} = 1$. This is true, since $\omega(-I) = 1$. However, condition (i) is

$$||A + B||_v = ||I - I||_1 = ||0|| = 0$$

But

$$2\max\{||A||_v, ||B||_v\} = 2\max\{||I||_1, ||-I||_1\} = 2\max\{1, 1\} = 2$$

Thus $0 \neq 2$, so the theorem fails.

Theorem 6.10. *If $A, B \in \mathcal{B}(\mathcal{H})$, $v \in [0, 1]$, and $M_v(B^*)M_v(A) = 0$, then the following conditions are equivalent:*

- *(i)* $||A + B||_v^2 = ||A||_v^2 + ||B||_v^2$
- *(ii) There exists a sequence of unit vectors $\{x_n\}$ in $\mathcal{H}$ such that*

$$\lim_{n\to\infty} \langle M_v(A^*)M_v(A)x_n, M_v(B^*)M_v(B)x_n\rangle = ||A||_v^2 ||B||_v{}^2$$

Proof of Theorem 6. $(i) \Rightarrow (ii)$: Because $M_v(B^*)M_v(A) = 0$, it follows that $(M_v(A))^*M_v(B) = 0$. Furthermore, by Theorem 1.3:

$$\begin{aligned} ||A + B||_v^2 &= ||M_v(A + B)||^2 = ||M_v(A + B)^*M_v(A + B)|| \\ &= ||(M_v(A) + M_v(B))^*(M_v(A) + M_v(B))|| \\ &= ||(M_v(A^*) + M_v(B^*))(M_v(A) + M_v(B))|| \\ &= ||M_v(A^*)M_v(A) + M_v(A^*)M_v(B) + M_v(B^*)M_v(A) + M_v(B^*)M_v(B)|| \\ &= ||M_v(A^*)M_v(A) + M_v(B^*)M_v(B)|| \end{aligned} \tag{21}$$

We also know $||M_v(A^*)M_v(A)|| = ||M_v(A)^*M_v(A)|| = ||M_v(A)||^2 = ||A||_v^2$. So, $||A + B||_v^2 \leq ||M_v(A^*)M_v(A)|| + ||M_v(B^*)M_v(B)|| = ||A||_v^2 + ||B||_v^2$. Given $||A + B||_v^2 = ||A||_v^2 + ||B||_v^2$, we must have

$$||M_v(A^*)M_v(A) + M_v(B^*)M_v(B)|| = ||M_v(A^*)M_v(A)|| + ||M_v(B^*)M_v(B)||$$

From Corollary 2 (the case for $v = 1$), there exists a unit sequence $\{x_n\}$ such that

$$\begin{aligned} &\lim_{n\to\infty} \langle (M_v(A))^*M_v(A)x_n, (M_v(B))^*M_v(B)x_n\rangle \\ &= ||(M_v(A))^*M_v(A)|| \cdot ||(M_v(B))^*M_v(B)|| \\ &= ||M_v(A)||^2||M_v(B)||^2 = ||A||_v^2||B||_v{}^2 \end{aligned}$$

$(ii) \Rightarrow (i)$: If there exists a unit sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} \langle M_v(A^*)M_v(A)x_n, M_v(B^*)M_v(B)x_n\rangle = ||A||_v^2||B||_v^2$$

then

$$\lim_{n\to\infty} \langle M_v(A^*)M_v(A)x_n, M_v(B^*)M_v(B)x_n\rangle = ||M_v(A^*)M_v(A)|| \cdot ||M_v(B^*)M_v(B)||$$

By Theorem 4 (with $v = 1$), this implies

$$||M_v(A^*)M_v(A) + M_v(B^*)M_v(B)|| = ||M_v(A^*)M_v(A)|| + ||M_v(B^*)M_v(B)||$$

Using (21) and the fact that $M_v(B^*)M_v(A) = 0$, this becomes

$$||A + B||_v^2 = ||A||_v^2 + ||B||_v^2$$

□

Fix $v \in [0, 1]$ and $A \in \mathcal{B}(\mathcal{H})$. Define the function $f : \mathbb{R} \to \mathbb{R}_+$

$$\theta \mapsto ||ve^{i\theta}A + (1 - v)e^{-i\theta}A^*||$$

It is known that f is a composition of continuous functions, so f is continuous. Also, $\mathrm{Ran}(f) = \mathrm{Ran}(f|_{[0,2\pi]})$. Thus, by the property that a continuous function on a compact set attains its maximum, we have the following direct fact:

$$\sup_{\theta\in\mathbb{R}} f(\theta) = \sup_{\theta\in[0,2\pi]} f(\theta) = \max_{\theta\in[0,2\pi]} f(\theta) \tag{22}$$

Theorem 6.11. *If $A, B \in \mathcal{B}(\mathcal{H})$ and $v \in [0, 1]$, then the following conditions are equivalent:*

- *(i) $\omega_v(A + B) = \omega_v(A) + \omega_v(B)$*
- *(ii) There exists $\theta \in [0, 2\pi]$ and a sequence of unit vectors $\{x_n\}$ in $\mathcal{H}$ such that*

$$\lim_{n\to\infty} \langle M_v(e^{i\theta}A)x_n, M_v(e^{i\theta}B)x_n\rangle = \omega_v(A)\omega_v(B)$$

Proof of Theorem 7. (*i*) ⇒ (*ii*): Since $\omega_v(A + B) = \sup_{\theta\in\mathbb{R}} ||M_v(e^{i\theta}(A + B))||$, from (22) there exists $\theta_0 \in [0, 2\pi]$ such that

$$||ve^{i\theta_0}(A + B) + (1 - \nu)e^{-i\theta_0}(A + B)^*|| = \omega_v(A + B)$$

By Theorem 1.1, there exists a unit sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} ||(ve^{i\theta_0}(A + B) + (1 - v)e^{-i\theta_0}(A + B)^*)x_n|| = \omega_\nu(A + B) \tag{23}$$

Also, by the definition of $\omega_v(A)$ and the triangle inequality:

$$\begin{aligned} ||M_v(e^{i\theta_0}(A + B))x_n|| &= ||M_v(e^{i\theta_0}A)x_n + M_v(e^{i\theta_0}B)x_n|| \\ &\le ||M_v(e^{i\theta_0}A)x_n|| + ||M_v(e^{i\theta_0}B)x_n|| \\ &\le ||M_v(e^{i\theta_0}A)|| + ||M_v(e^{i\theta_0}B)|| \\ &\le \omega_v(A) + \omega_v(B) \end{aligned}$$

Given $\omega_v(A+B) = \omega_v(A) + \omega_v(B)$, and taking the limit $n \to \infty$ of the above using (23), all inequalities must be equalities. This implies:

$$\lim_{n\to\infty} ||(ve^{i\theta_0}A + (1-v)e^{-i\theta_0}A^*)x_n|| = \omega_v(A) \tag{24}$$

and

$$\lim_{n\to\infty} ||(ve^{i\theta_0}B + (1-v)e^{-i\theta_0}B^*)x_n|| = \omega_v(B) \tag{25}$$

On the other hand, expanding the squared norm:

$$\begin{aligned}
||M_v(e^{i\theta_0}(A+B))x_n||^2 &= ||M_v(e^{i\theta_0}A)x_n + M_v(e^{i\theta_0}B)x_n||^2 \\
&= ||M_v(e^{i\theta_0}A)x_n||^2 + ||M_v(e^{i\theta_0}B)x_n||^2 + 2\mathcal{R}\langle M_v(e^{i\theta_0}A)x_n, M_v(e^{i\theta_0}B)x_n\rangle
\end{aligned}$$

Taking the limit $n \to \infty$ and using (23), (24), and (25):

$$(\omega_v(A) + \omega_v(B))^2 = \omega_v(A)^2 + \omega_v(B)^2 + 2\lim_{n\to\infty} \mathcal{R}\langle M_v(e^{i\theta_0}A)x_n, M_v(e^{i\theta_0}B)x_n\rangle$$

This implies $\lim_{n\to\infty} \mathcal{R}\langle M_v(e^{i\theta_0}A)x_n, M_v(e^{i\theta_0}B)x_n\rangle = \omega_v(A)\omega_v(B)$. By Cauchy-Schwarz, $\lim_{n\to\infty} |\langle M_v(e^{i\theta_0}A)x_n, M_v(e^{i\theta_0}B)x_n\rangle| \leq \omega_v(A)\omega_v(B)$. Since the real part attains this limit, by Lemma 2, we have

$$\lim_{n\to\infty} \langle M_v(e^{i\theta_0}A)x_n, M_v(e^{i\theta_0}B)x_n\rangle = \omega_v(A)\omega_v(B)$$

□

Continuation of Theorem 7. *(ii)* ⇒ *(i)*: Assume there exists $\theta \in [0, 2\pi]$ and a unit sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} \langle M_v(e^{i\theta}A)x_n, M_v(e^{i\theta}B)x_n\rangle = \omega_v(A)\omega_v(B) \tag{26}$$

By the Cauchy-Schwarz inequality and the definition of the weighted numerical radius, we have

$$\begin{aligned}
|\langle M_v(e^{i\theta}A)x_n, M_v(e^{i\theta}B)x_n\rangle| &\leq ||M_v(e^{i\theta}A)x_n|| \cdot ||M_v(e^{i\theta}B)x_n|| \\
&\leq ||M_v(e^{i\theta}A)x_n|| \cdot ||M_v(e^{i\theta}B)|| \\
&\leq ||M_v(e^{i\theta}A)x_n|| \cdot \omega_v(B) \\
&\leq ||M_v(e^{i\theta}A)|| \cdot \omega_v(B) \\
&\leq \omega_v(A)\omega_v(B)
\end{aligned}$$

Taking the limit $n \to \infty$, (26) implies all these inequalities must be equalities. Thus,

$$\omega_v(A) = ||M_v(e^{i\theta}A)|| \quad \text{and} \quad \omega_v(B) = ||M_v(e^{i\theta}B)||$$

and

$$\lim_{n\to\infty} \langle M_v(e^{i\theta}A)x_n, M_v(e^{i\theta}B)x_n\rangle = ||M_v(e^{i\theta}A)|| \cdot ||M_v(e^{i\theta}B)||$$

By Theorem 4, this implies

$$||M_v(e^{i\theta}(A+B))|| = ||M_v(e^{i\theta}A)|| + ||M_v(e^{i\theta}B)|| = \omega_v(A) + \omega_v(B)$$

By the definition of the weighted numerical radius and Theorem 2.2:

$$||M_v(e^{i\theta}(A+B))|| \leq \omega_v(A+B) \leq \omega_v(A) + \omega_v(B)$$

Therefore, $\omega_v(A+B) = \omega_v(A) + \omega_v(B)$. □

To characterize more equality conditions, we first turn to the study of a class of convex functions.

Theorem 6.12. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and $v \in [0,1]$. Define the function $f : \mathbb{R} \to \mathbb{R}_+$ that satisfies*

$$t \mapsto ||tA + (1-t)B||_v$$

Then f is a convex function on $\mathbb{R}$.

Proof. Let $\alpha \in (0,1)$, $\beta = 1-\alpha$, and $t_1, t_2 \in \mathbb{R}$.

$$\begin{aligned}
f(\alpha t_1 + \beta t_2) &= ||v((\alpha t_1 + \beta t_2)A + (1 - \alpha t_1 - \beta t_2)B) \\
&\qquad + (1-v)((\alpha t_1 + \beta t_2)A^* + (1 - \alpha t_1 - \beta t_2)B^*)|| \\
&= ||v(\alpha t_1 A + \beta t_2 A + B - \alpha t_1 B - \beta t_2 B) \\
&\qquad + (1-v)(\alpha t_1 A^* + \beta t_2 A^* + B^* - \alpha t_1 B^* - \beta t_2 B^*)|| \\
&= ||\alpha(v(t_1 A + (1-t_1)B) + (1-v)(t_1 A^* + (1-t_1)B^*)) \\
&\qquad + \beta(v(t_2 A + (1-t_2)B) + (1-v)(t_2 A^* + (1-t_2)B^*))|| \\
&\leq \alpha||v(t_1 A + (1-t_1)B) + (1-v)(t_1 A^* + (1-t_1)B^*)|| \\
&\qquad + \beta||v(t_2 A + (1-t_2)B) + (1-v)(t_2 A^* + (1-t_2)B^*)|| \\
&= \alpha f(t_1) + \beta f(t_2)
\end{aligned}$$

By Definition 1.2, this is proven. □

Corollary 6.13. *Let $A \in \mathcal{B}(\mathcal{H})$. Define the function $f : [0,1] \to \mathbb{R}_+$ that satisfies*

$$v \mapsto ||vA + (1-v)A^*||$$

Then f is a continuous convex function on $[0,1]$. f attains its minimum value at $v = \frac{1}{2}$ and its maximum value at $v = 0, 1$. Furthermore, f is monotonically decreasing on $[0, \frac{1}{2}]$ and monotonically increasing on $[\frac{1}{2}, 1]$.

Proof. Let $B = A^*$ and $v = 1$ in Theorem 8. This shows $t \mapsto ||tA + (1-t)A^*||$ is convex. So $f(v)$ is convex. Using the convexity of f and Theorem 1.3, we have

$$\begin{aligned}
f\left(\frac{1}{2}\right) &= f\left(\frac{v + (1-v)}{2}\right) \\
&\leq \frac{1}{2}f(v) + \frac{1}{2}f(1-v) \\
&= \frac{1}{2}||vA + (1-v)A^*|| + \frac{1}{2}||(1-v)A + vA^*|| \\
&= \frac{1}{2}||vA + (1-v)A^*|| + \frac{1}{2}||(vA + (1-v)A^*)^*|| \\
&= ||vA + (1-v)A^*|| = f(v)
\end{aligned}$$

Therefore, f attains its minimum value at $v = \frac{1}{2}$. By Theorem 1.3, $f(0) = ||A^*|| = ||A||$ and $f(1) = ||A||$, so $f(0) = f(1)$. By convexity, for $v \in [0,1]$:

$$f(v) = f(v \cdot 1 + (1-v) \cdot 0) \leq vf(1) + (1-v)f(0) = vf(1) + (1-v)f(1) = f(1) = f(0)$$

Thus, f attains its maximum value at $v = 0, 1$. Continuity is clear as f is a composition of continuous functions.

For symmetry, let $v \in [0, \frac{1}{2}]$.

$$\begin{aligned} f\left(\frac{1}{2} - v\right) &= ||\left(\frac{1}{2} - v\right)A + \left(1 - \frac{1}{2} + v\right)A^*|| \\ &= ||\left(\frac{1}{2} - v\right)A + \left(\frac{1}{2} + v\right)A^*|| \\ &= ||\left(\left(\frac{1}{2} - v\right)A^* + \left(\frac{1}{2} + v\right)A\right)^*|| \\ &= ||\left(\frac{1}{2} + v\right)A + \left(\frac{1}{2} - v\right)A^*|| = f\left(\frac{1}{2} + v\right) \end{aligned}$$

Thus, f is symmetric about $v = \frac{1}{2}$. Let $v_1 < v_2 < \frac{1}{2}$. By Lemma 1.1 (on convexity),

$$\frac{f(v_2) - f(v_1)}{v_2 - v_1} \leq \frac{f(\frac{1}{2}) - f(v_2)}{\frac{1}{2} - v_2}$$

Since $f(\frac{1}{2})$ is the minimum, the right side is ≤ 0. Thus, $\frac{f(v_2)-f(v_1)}{v_2-v_1} \leq 0$, which implies $f(v_2) \leq f(v_1)$. Therefore, f is monotonically decreasing on $[0, \frac{1}{2}]$ and, by symmetry, monotonically increasing on $[\frac{1}{2}, 1]$. □

Remark 6. From Corollary 4, we can assert that for any $A \in \mathcal{B}(\mathcal{H})$, the function $f_A(v) = \omega_v(A)$ is a convex function on $[0,1]$. This is because $f_A(v) = \sup_{\theta \in \mathbb{R}} ||ve^{i\theta}A + (1-v)e^{-i\theta}A^*||$. Let $A' = e^{i\theta}A$ in Corollary 4. We know $v \mapsto ||ve^{i\theta}A + (1-v)e^{-i\theta}A^*||$ is convex for any fixed θ. Let $f_\theta(v) = ||ve^{i\theta}A + (1-v)e^{-i\theta}A^*||$. For $\alpha + \beta = 1$ and $v_1, v_2 \in [0,1]$:

$$f_\theta(\alpha v_1 + \beta v_2) \leq \alpha f_\theta(v_1) + \beta f_\theta(v_2)$$

Taking the supremum over $\theta \in \mathbb{R}$ on both sides:

$$\begin{aligned} f_A(\alpha v_1 + \beta v_2) &= \sup_{\theta \in \mathbb{R}} f_\theta(\alpha v_1 + \beta v_2) \\ &\leq \sup_{\theta \in \mathbb{R}} (\alpha f_\theta(v_1) + \beta f_\theta(v_2)) \\ &\leq \alpha \sup_{\theta \in \mathbb{R}} f_\theta(v_1) + \beta \sup_{\theta \in \mathbb{R}} f_\theta(v_2) \\ &= \alpha f_A(v_1) + \beta f_A(v_2) \end{aligned}$$

Thus, $f_A(v) = \omega_v(A)$ is a convex function. We wish to point out that the difference of convex functions induced by operators is not necessarily a convex function.

Example 6.2. Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$.

$$\begin{aligned} \omega_v(A) &= \sup_{\theta \in \mathbb{R}} ||ve^{i\theta}A + (1-v)e^{-i\theta}A^*|| \\ &= \sup_{\theta \in \mathbb{R}} \left\| \begin{pmatrix} 0 & ve^{i\theta} \\ (1-v)e^{-i\theta} & 0 \end{pmatrix} \right\| \end{aligned}$$

The norm of this matrix is the square root of the largest eigenvalue of M^*M, where $M = \begin{pmatrix} 0 & ve^{i\theta} \\ (1-v)e^{-i\theta} & 0 \end{pmatrix}$.

$$M^*M = \begin{pmatrix} (1-v)^2 & 0 \\ 0 & v^2 \end{pmatrix}$$

The eigenvalues are $(1-v)^2$ and v^2.

$$\omega_v(A) = \sup_{\theta \in \mathbb{R}} \sqrt{\max\{(1-v)^2, v^2\}} = \max\{v, 1-v\}$$

Similarly, $\omega_v(B) = \max\{2v, 2(1-v)\}$. Then, let $f(v) = \omega_v(A) - \omega_v(B)$.

$$f(v) = \max\{v, 1-v\} - \max\{2v, 2(1-v)\} = \begin{cases} (1-v) - 2(1-v) = v - 1 & v \in [0, \frac{1}{2}] \\ v - 2v = -v & v \in [\frac{1}{2}, 1] \end{cases}$$

This function is evidently concave.

Corollary 6.14. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and $v \in [0,1]$. Then*

$$\begin{aligned} ||A+B||_v &\le (1-\lambda)||(1+\lambda)A + (1-\lambda)B||_v + \lambda||\lambda A + (2-\lambda)B||_v \\ &\le 2\int_0^1 ||tA + (1-t)B||_v dt \\ &\le ||\lambda A + (1-\lambda)B||_v + (1-\lambda)||A||_v + \lambda||B||_v \\ &\le ||A||_v + ||B||_v \end{aligned}$$

Proof. By Theorem 8, $f(t) = ||tA + (1-t)B||_v$ is a convex function. By Theorem 2.3 (Hadamard-Hammer-Bullen), setting $x = 0, y = 1$:

$$\begin{aligned} f\left(\frac{1}{2}\right) &\le (1-\lambda)f\left(\frac{1+\lambda}{2}\right) + \lambda f\left(\frac{\lambda}{2}\right) \\ &\le \int_0^1 f(t)dt \\ &\le \frac{1}{2}(f(\lambda) + (1-\lambda)f(1) + \lambda f(0)) \\ &\le \frac{f(0) + f(1)}{2} \end{aligned}$$

Substituting the definitions $f(0) = ||B||_v$, $f(1) = ||A||_v$, $f(\frac{1}{2}) = ||\frac{1}{2}A + \frac{1}{2}B||_v$, etc., gives the result. (Note: The first and last lines of the corollary seem to be scaled by 2, e.g., $2f(1/2) \le ... \le f(0) + f(1)$). □

Remark 7. Through the direct relationship below, we establish the subsequent results. For any $A \in \mathcal{B}(\mathcal{H})$, its weighted numerical radius $\omega_v(A)$ and weighted operator norm $||A||_v$ satisfy

$$\omega_v(A) = \sup_{\theta\in\mathbb{R}} ||e^{i\theta}A||_v \tag{27}$$

Corollary 6.15. *If $A, B \in \mathcal{B}(\mathcal{H})$ and $v \in [0, 1]$, then*

$$\begin{aligned}\omega_v(A+B) &\le \sup_{\theta\in\mathbb{R}} \big((1-\lambda)||(1+\lambda)e^{i\theta}A + (1-\lambda)e^{i\theta}B||_v + \lambda||\lambda e^{i\theta}A + (2-\lambda)e^{i\theta}B||_v\big) \\ &\le \sup_{\theta\in\mathbb{R}} 2\int_0^1 ||te^{i\theta}A + (1-t)e^{i\theta}B||_v dt \\ &\le \sup_{\theta\in\mathbb{R}} \big(||\lambda e^{i\theta}A + (1-\lambda)e^{i\theta}B||_v + (1-\lambda)||e^{i\theta}A||_v + \lambda||e^{i\theta}B||_v\big) \\ &\le \omega_v(A) + \omega_v(B)\end{aligned}$$

Proof. Let $A = e^{i\theta}A$ and $B = e^{i\theta}B$ in Corollary 5. By (27), taking the supremum with respect to $\theta \in \mathbb{R}$ proves the result. □

Example 6.3. Let $A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then

$$\omega_v(A) = \sup_{\theta\in\mathbb{R}} ||ve^{i\theta}A + (1-v)e^{-i\theta}A^*|| = \sup_{\theta\in\mathbb{R}} \left\| \begin{pmatrix} 0 & (1-v)e^{-i\theta} & 0 \\ ve^{i\theta} & 0 & (1-v)e^{-i\theta} \\ 0 & ve^{i\theta} & 0 \end{pmatrix} \right\|$$

The norm is $\sqrt{r(M^*M)}$. $M^*M = \begin{pmatrix} v^2 & 0 & v(1-v)e^{-2i\theta} \\ 0 & (1-v)^2+v^2 & 0 \\ v(1-v)e^{2i\theta} & 0 & (1-v)^2 \end{pmatrix}$ By Theorem 1.7, the norm is the max eigenvalue: $\lambda_{max} = \max\{\lambda : \det(M^*M - \lambda I) = 0\}$ $\det = (\lambda - ((1-v)^2 + v^2)) \cdot [(\lambda - v^2)(\lambda - (1-v)^2) - v^2(1-v)^2] = 0$ $\det = (\lambda - (1 - 2v + 2v^2)) \cdot [\lambda^2 - \lambda + v^2(1-v)^2 - v^2(1-v)^2] = 0$ $\det = (\lambda - (1 - 2v + 2v^2)) \cdot \lambda(\lambda - 1) = 0$. The eigenvalues are $0, 1, 1 - 2v + 2v^2$. $\omega_v(A) = \sqrt{\max\{1, 1 - 2v + 2v^2\}} = \sqrt{1 - 2v + 2v^2}$ (since $1 - 2v + 2v^2 \ge 1/2$).

Similarly,

$$\omega_v(B) = \sup_{\theta\in\mathbb{R}} \left\| \begin{pmatrix} 0 & ve^{i\theta} & ve^{i\theta} \\ (1-v)e^{-i\theta} & 0 & 0 \\ (1-v)e^{-i\theta} & 0 & 0 \end{pmatrix} \right\|$$

$M^*M = \begin{pmatrix} 2(1-v)^2 & 0 & 0 \\ 0 & v^2 & v^2 \\ 0 & v^2 & v^2 \end{pmatrix}$ The eigenvalues are $2(1-v)^2, 2v^2, 0$. $\omega_v(B) = \sqrt{\max\{2(1-v)^2, 2v^2\}} = \sqrt{2}\max\{v, 1-v\}$. Thus, $\omega_v(A+B) \le \sqrt{1-2v+2v^2} + \sqrt{2}\max\{v, 1-v\}$. In particular, when $v = \frac{1}{2}$:

$$\omega(A+B) = \omega_{\frac{1}{2}}(A+B) \le \sqrt{1-1+1/2} + \sqrt{2}(\frac{1}{2}) = \sqrt{\frac{1}{2}} + \frac{\sqrt{2}}{2} = \sqrt{2}$$

$A + B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. This is the companion matrix for $p(z) = z^3 - z - 1$. For $p(z) = z^3 + z + 1$, the companion matrix is $C(p) = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$. The modulus of the zeros λ of $p(z) = z^3 + z + 1$ satisfies $|\lambda| \leq \sqrt{2}$. On the other hand, Carmichael and Mason [12] have the result: $|\lambda| \leq \sqrt{\sum_{i=0}^{n-1} |a_i|^2 + 1}$. For $p(z) = z^3 + z + 1$, this gives $|\lambda| \leq \sqrt{|1|^2 + |1|^2 + 1} = \sqrt{3}$. Thus, our result $|\lambda| \leq \sqrt{2}$ strengthens the result of Carmichael and Mason. In fact, the roots are $z_1 \approx -0.682$, $z_2 \approx 0.341 - 1.162i$, $z_3 \approx 0.341 + 1.162i$. $(|z_2|, |z_3| \approx 1.21)$.

Corollary 6.16. *If $A \in \mathcal{B}(\mathcal{H})$, $v \in [0, 1]$, and $\alpha = \max\{v, 1 - v\}$, then*

$$\frac{1}{2\alpha}\omega_v(A^2) \leq \omega(A^2) \leq \omega^2(A) \leq \omega_v^2(A)$$

Proof. From Corollary 4, we know $f(v) = ||e^{i\theta}A||_v$ is convex and minimized at $v = 1/2$, so $||e^{i\theta}A||_{\frac{1}{2}} \leq ||e^{i\theta}A||_v$. Taking the supremum over θ and using (2) and (27), we have

$$\omega(A) = \omega_{\frac{1}{2}}(A) \leq \omega_v(A)$$

Combining this with Theorem 2.4 (which states $\omega_v(A) \leq 2\alpha\omega(A)$ for $v \in [1/2, 1]$... Note: The proof seems to use $\omega_v(A^2) \leq 2\alpha\omega(A^2)$) and Theorem 2.1 ($\omega(A^2) \leq \omega^2(A)$) gives:

$$\frac{1}{2\alpha}\omega_v(A^2) \leq \omega(A^2) \leq \omega^2(A) \leq \omega_v^2(A)$$

□

Theorem 6.17. *If $A, B \in \mathcal{B}(\mathcal{H})$ and $v \in [0, 1]$, then the following conditions are equivalent:*

- *(i)* $\omega_v(AB) = \max\{||A||^2, ||B||^2\}$
- *(ii) There exists a sequence of unit vectors $\{x_n\}$ in $\mathcal{H}$ such that*

$$\lim_{n\to\infty} |\langle ABx_n, (AB)^* x_n\rangle| = \max\{||A||^4, ||B||^4\}$$

Proof. $(i) \Rightarrow (ii)$: From Definition 9,

$$\omega_v(AB) = \sup_{\theta\in\mathbb{R}} ||ve^{i\theta}AB + (1 - \nu)e^{-i\theta}(AB)^*|| = \max\{||A||^2, ||B||^2\}$$

Combining with (22), there exists θ_0 such that $||M_\nu(e^{i\theta_0}AB)|| = \max\{||A||^2, ||B||^2\}$. Furthermore,

$$\begin{aligned} \max\{||A||^2, ||B||^2\} &= ||M_\nu(e^{i\theta_0}AB)|| \\ &= ||ve^{i\theta_0}AB + (1 - v)e^{-i\theta_0}(AB)^*|| \\ &\leq v||AB|| + (1 - v)||(AB)^*|| = ||AB|| \\ &\leq ||A||||B|| \leq \frac{||A||^2 + ||B||^2}{2} \leq \max\{||A||^2, ||B||^2\} \end{aligned}$$

All inequalities must be equalities. This implies $||A|| = ||B||$ and $||AB||_v = ||A||||B||$. By Theorem 2, there exists a unit sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} \langle (e^{i\theta_0}AB)x_n, ((e^{i\theta_0}AB))^* x_n\rangle = ||e^{i\theta_0}A||^2||B||^2 = ||A||^2||B||^2 = \max\{||A||^4, ||B||^4\}$$

Thus, $\lim_{n\to\infty} |\langle ABx_n, (AB)^* x_n\rangle| = \max\{||A||^4, ||B||^4\}$.

$(ii) \Rightarrow (i)$: Assume there exists a unit sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} |\langle ABx_n, (AB)^* x_n\rangle| = \max\{||A||^4, ||B||^4\}$$

Then, by Definition 9, Corollary 7, and (27), we have the chain of inequalities:

$$\begin{aligned}\max\{||A||^4, ||B||^4\} &\le \omega((AB)^2) \quad \text{(This limit is } \omega((AB)^2) \text{ if } AB \text{ is normal)}\\ &\le \omega^2(AB) \quad \text{(by Theorem 2.1)}\\ &\le {\omega_v}^2(AB) \quad \text{(by Corollary 7 proof)}\\ &\le ||AB||^2 \le (||A||||B||)^2 \le (\max\{||A||^2, ||B||^2\})^2 = \max\{||A||^4, ||B||^4\}\end{aligned}$$

All inequalities must be equalities. Thus, $\omega_v^2(AB) = \max\{||A||^4, ||B||^4\}$, which means $\omega_v(AB) = \max\{||A||^2, ||B||^2\}$. □

Remark 8. From Corollary 6 and Corollary 7, for any $A, B \in \mathcal{B}(\mathcal{H})$, $v \in [0,1]$, $\alpha = \max\{v, 1-v\}$, it holds that

$$\omega_v(A^2 + B^2) \le \omega_v(A^2) + \omega_v(B^2) \le 2\alpha\omega_v^2(A) + 2\alpha\omega_v^2(B) \le 4\alpha \max\{\omega_v^2(A), \omega_v^2(B)\}$$

Theorem 6.18. *If $A, B \in \mathcal{B}(\mathcal{H})$, $v \in [0,1]$, $\alpha = \max\{v, 1-v\}$, then the following conditions are equivalent:*

- *(i)* $\omega_v(A^2 + B^2) = 4\alpha \max\{\omega_v^2(A), \omega_v^2(B)\}$
- *(ii) There exists $\theta \in [0, 2\pi]$ and a sequence of unit vectors $\{x_n\}$ in $\mathcal{H}$ such that*

$$\lim_{n\to\infty} \langle M_v(e^{i\theta}A^2)x_n, M_v(e^{i\theta}B^2)x_n\rangle = 4\alpha^2 \max\{\omega_v^4(A), \omega_v^4(B)\}$$

Proof of Theorem 10. $(i) \Rightarrow (ii)$: From the facts preceding the theorem, equality must hold throughout the chain. This means:

$$\omega_v(A^2 + B^2) = \omega_v(A^2) + \omega_v(B^2)$$

and

$$\omega_v(A^2) = 2\alpha\omega_v^2(A) \quad \text{and} \quad \omega_v(B^2) = 2\alpha\omega_v^2(B)$$

and

$$\omega_v^2(A) = \omega_v^2(B) = \max\{\omega_v^2(A), \omega_v^2(B)\}$$

From $\omega_v(A^2 + B^2) = \omega_v(A^2) + \omega_v(B^2)$, by Theorem 7, there exists $\theta \in [0, 2\pi]$ and a unit sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} \langle M_v(e^{i\theta}A^2)x_n, M_v(e^{i\theta}B^2)x_n\rangle = \omega_v(A^2)\omega_v(B^2)$$

Substituting the equalities from above:

$$= (2\alpha\omega_v^2(A))(2\alpha\omega_v^2(B)) = 4\alpha^2 \max\{\omega_v^4(A), \omega_v^4(B)\}$$

$(ii) \Rightarrow (i)$: By Lemma 1, the condition implies

$$\lim_{n\to\infty} \mathcal{R}\langle M_v(e^{i\theta}A^2)x_n, M_v(e^{i\theta}B^2)x_n\rangle = 4\alpha^2 \max\{\omega_v^4(A), {\omega_v}^4(B)\} \tag{28}$$

Furthermore, by the Cauchy inequality and Corollary 7:

$$\begin{aligned}
\mathcal{R}\langle M_v(e^{i\theta}A^2)x_n, \dots\rangle &\leq |\langle M_v(e^{i\theta}A^2)x_n, M_v(e^{i\theta}B^2)x_n\rangle| \\
&\leq ||M_v(e^{i\theta}A^2)|| \cdot ||M_v(e^{i\theta}B^2)|| \\
&\leq \omega_v(A^2)\omega_v(B^2) \\
&\leq (2\alpha\omega_v^2(A))(2\alpha\omega_v^2(B)) \\
&\leq 4\alpha^2 \frac{\omega_v^4(A) + \omega_v^4(B)}{2} \leq 4\alpha^2 \max\{\omega_v^4(A), \omega_v^4(B)\}
\end{aligned}$$

Taking the limit $n \to \infty$, (28) implies all inequalities must be equalities. This gives:

$$\omega_v(A^2)\omega_v(B^2) = 4\alpha^2 \max\{\omega_v^4(A), \omega_v^4(B)\}$$

and (combined with Corollary 7)

$$\omega_v(A^2) = 2\alpha\omega_v^2(A) \quad \text{and} \quad \omega_v(B^2) = 2\alpha\omega_v^2(B) \quad \text{and} \quad \omega_v^2(A) = \omega_v^2(B) \tag{29}$$

Therefore, $\lim_{n\to\infty}\langle M_\nu(e^{i\theta}A^2)x_n, M_\nu(e^{i\theta}B^2)x_n\rangle = \omega_\nu(A^2)\omega_\nu(B^2)$. By Theorem 7, this implies

$$\omega_v(A^2 + B^2) = \omega_v(A^2) + \omega_v(B^2)$$

Using (29), we get

$$\omega_v(A^2 + B^2) = 2\alpha\omega_v^2(A) + 2\alpha\omega_v^2(B) = 4\alpha \max\{\omega_v^2(A), \omega_v^2(B)\}$$

□

References

[1] Conde C, Sababheh M, Moradi H R. Some weighted numerical radius inequalities[J]. arxiv preprint arxiv:2204.07620, 2022.

[2] Sheikhhosseini A, Khosravi M, Sababheh M. The weighted numerical radius[J]. Annals of Functional Analysis, 2022, 13(1): 3.

[3] Zhan Xingzhi. *Matrix Theory*. Higher Education Press.

[4] Xu Quanhua, Ma Tao, Yin Zhi. *Lectures on Functional Analysis*. Higher Education Press.

[5] Guo Kunyu. *Foundations of Operator Theory*. Fudan University Press.

[6] Bhunia P, Dragomir S S, Moslehian M S, et al. *Lectures on numerical radius inequalities* [M]. Cham: Springer, 2022.

[7] Yamazaki T. On upper and lower bounds of the numerical radius and an equality condition[J]. Studia mathematica, 2007, 1(178): 83-89.

[8] Conway J B. *A course in functional analysis* [M]. Springer, 2019.

[9] Sahoo S, Moradi H, Sababheh M. A mixed Cauchy-Schwarz inequality with applications[J]. 2023.

[10] Chen Jixiu, Yu Chonghua, Jin Lu. *Mathematical Analysis*. Higher Education Press.

[11] Chandra Rout N, Sahoo S, Mishra D. Some A-numerical radius inequalities for semi-Hilbertian space operators[J]. Linear and Multilinear Algebra, 2021, 69(5): 980-996.

[12] Horn R A, Johnson C R. *Matrix analysis* [M]. Cambridge university press, 2012.

Acknowledgments

1. Source of the research topic and background: This paper is mainly influenced by references [1] and [2]. These two articles recently defined the weighted numerical radius of operators in different ways. We are interested in the differences and unifications of these two definitions, as well as their connection to the classical numerical radius. Regarding the definition of the numerical radius in reference [2]—that is, for any bounded linear operator A on a Hilbert space $\mathcal{H}$ and $v \in [0, 1]$,

$$\omega_v(A) = \sup_{\theta \in \mathbb{R}} ||ve^{i\theta} A + (1-v)e^{-i\theta} A^*||$$

is the weighted numerical radius of the operator A. We were inspired by the definition of the weighted operator norm in [1] and, based on [2], provide a new definition of the weighted operator norm. Specifically, we call

$$M_v(A) = vA + (1-v)A^*$$

the weighted average transformation of the operator A, and denote

$$||A||_v \triangleq ||M_v(A)||$$

as the weighted operator norm of A. When $v = \frac{1}{2}$, $\omega_v(A) = \omega(A)$, and when $v = 1$, $||A||_v = ||A||$. Thus, our results indeed hold for the classical numerical radius and norm. A major application of the numerical radius is to estimate the roots of polynomials, which can be seen in Example 3. Therefore, we utilize the convexity of the weighted numerical radius combined with the Hadamard inequality to develop estimates for the numerical radius, particularly focusing on inequalities of the weighted numerical radius. Naturally, the conditions for

equality in these inequalities are also worth noting. We establish necessary and sufficient conditions for some boundary equalities such as:

$$\begin{aligned}
\omega(M_v(B^*)M_v(A)) &= \max\{||A||_v^2, ||B||_v^2\} \\
\omega_v(AB) &= \max\{||A||^2, ||B||^2\} \\
\omega_v(A^2+B^2) &= 4\alpha \max\{\omega_v^2(A), \omega_v^2(B)\} \\
||A+B||_v &= 2\max\{||A||_v, ||B||_v\}
\end{aligned}$$

The necessary and sufficient conditions for the Pythagorean theorem for the weighted operator norm,

$$||A+B||_v{}^2 = ||A||_v{}^2 + ||B||_v{}^2$$

are also provided in Theorem 6. We also established the necessary and sufficient conditions for the following equalities to hold:

$$\begin{aligned}
||A||_v &= ||A|| \\
||A+B||_v &= ||A||_v + ||B||_v \\
||A+B||_v &= 2\max\{||A||_v, ||B||_v\}
\end{aligned}$$

In particular, we use Example 3 to show that Corollary 6 strengthens the results of Carmichael and Mason.

2. The work and contributions of each team member in the writing of the paper: The team members shared the work in writing the paper, discussed the topic together, and conducted calculations and proofs. Both members played indispensable roles in the team.

3. The relationship between the advisor and the students, and the role it plays in the writing process: The advisor provided guidance on the selection of the topic, the direction, and the structure of the paper, leading the overall framework and content of the paper. Is the guidance compensated: Unpaid.

4. Research results completed with the assistance of others: None.

5. The main difficulties encountered: When familiarizing ourselves with operator theory, it was necessary to have a certain knowledge of linear algebra and to be proficient in the transformations of norms and inner products. This took some time to learn and prepare. When facing the calculation of the weighted numerical radius, we utilized some computational software to assist in the calculations, ensuring the correctness of the results.

The author thanks the S. T. Yau High School Science Award for providing a platform and opportunity to fully showcase themselves.