Will's Linear Algebra Project Session 1

UWCCSC Math Circle

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Worksheet: Answer Key

Problems

Part 1: Linear Independence

1. **Vector Space:** $\mathbb{R}^3$, Set $S_1 = \{v_1, v_2, v_3\}$
   Answer: Linearly Dependent.
   To test, we set $c_1v_1 + c_2v_2 + c_3v_3 = \mathbf{0}$:

   $$c_1(1,0,1) + c_2(1,2,0) + c_3(0,-2,1) = (0,0,0)$$

   This gives the system of equations:

   $$\begin{aligned} c_1 + c_2 \quad &= 0 \\ 2c_2 - 2c_3 &= 0 \implies c_2 = c_3 \\ c_1 + c_3 &= 0 \end{aligned}$$

   From the first equation, $c_1 = -c_2$. Substituting into the third equation: $(-c_2) + c_3 = 0 \implies c_3 = c_2$. This is the same as the second equation. Since the system reduces to $c_1 = -c_2$ and $c_3 = c_2$, it has non-trivial solutions. For example, let $c_2 = 1$. Then $c_1 = -1$ and $c_3 = 1$. We can verify: $(-1)v_1 + (1)v_2 + (1)v_3 = \mathbf{0}$. Thus, the set is dependent.

2. **Vector Space:** $\mathcal{P}_2(\mathbb{R})$, Set $S_2 = \{p_1, p_2, p_3\}$
   Answer: Linearly Dependent.
   We check if one vector is a combination of the others. Let's try to build p_3 from p_1 and p_2. Does $p_3 = c_1p_1 + c_2p_2$?

   $$x^2 + x - 1 = c_1(x^2+1) + c_2(x-2)$$
   $$x^2 + x - 1 = c_1x^2 + c_2x + (c_1 - 2c_2)$$

   By comparing coefficients:

   - x^2: $c_1 = 1$
   - x: $c_2 = 1$
   - constant: $c_1 - 2c_2 = 1 - 2(1) = -1$. This matches.

   Since $p_3 = p_1 + p_2$, we have a non-trivial relationship: $p_1 + p_2 - p_3 = \mathbf{0}$. The set is dependent.

3. **Vector Space:** $\mathbb{C}^2$, Set $S_4 = \{v_1, v_2\}$
   Answer: Linearly Independent.
   To test, we set $c_1v_1 + c_2v_2 = \mathbf{0}$, where $c_1, c_2 \in \mathbb{C}$:

   $$c_1(1,1) + c_2(1,i) = (0,0)$$

   This gives the system:

   $$\begin{aligned} c_1 + c_2 = 0 &\implies c_1 = -c_2 \\ c_1 + ic_2 = 0 & \end{aligned}$$

   Substitute the first equation into the second:

   $$(-c_2) + ic_2 = 0 \implies c_2(-1+i) = 0$$

   Since $(-1+i) \neq 0$, we can divide by it to get $c_2 = 0$. Substituting back, $c_1 = -0 = 0$. The only solution is the trivial solution $c_1 = 0, c_2 = 0$. Thus, the set is independent.

Part 2: Basis and Dimension

1. **Vector Space:** $\mathbb{R}^2$
   Answer: Yes, it is a basis.
   The dimension of $\mathbb{R}^2$ is 2. Since our set has 2 vectors, we only need to check for linear independence.

   $$c_1(1,2) + c_2(3,1) = (0,0)$$

   This gives the system:

   $$\begin{aligned} c_1 + 3c_2 = 0 &\implies c_1 = -3c_2 \\ 2c_1 + c_2 = 0 & \end{aligned}$$

   Substitute: $2(-3c_2) + c_2 = 0 \implies -6c_2 + c_2 = 0 \implies -5c_2 = 0 \implies c_2 = 0$. This implies $c_1 = -3(0) = 0$. Since the only solution is $c_1 = 0, c_2 = 0$, the set is linearly independent. An independent set of 2 vectors in a 2-dimensional space is a basis.

2. **Vector Space:** $\mathbb{R}^3$ (Subspace $W = \{(x,y,z) \mid x + y - z = 0\}$)
   (a) **Basis:** We find the general form of a vector in W. The constraint is $z = x + y$. Any vector $w \in W$ can be written as:

   $$w = (x,y,z) = (x,y,x+y)$$

   We can decompose this vector by its free variables (x and y):

   $$w = (x,0,x) + (0,y,y)$$

   $$w = x(1,0,1) + y(0,1,1)$$

   This shows that every vector in W is a linear combination of $\{(1,0,1),(0,1,1)\}$. This set spans W. By inspection, the two vectors are not multiples of each other, so they are linearly independent. A basis for W is $B_W = \{(1,0,1),(0,1,1)\}$. (Other answers are possible).
   (b) **Dimension:** The basis contains 2 vectors. Therefore, $\dim(W) = 2$.

3. **Vector Space:** $\mathcal{P}_3(\mathbb{R})$
   (a) **Basis:** A standard basis for $\mathcal{P}_3(\mathbb{R})$ must be able to generate any polynomial ax^3+bx^2+cx+d. The simplest such basis is $B = \{1, x, x^2, x^3\}$.
   (b) **Dimension:** The basis contains 4 vectors. Therefore, $\dim(\mathcal{P}_3(\mathbb{R})) = 4$.