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度量空间:(集合+距离)

$\Downarrow$

赋范空间 (向量空间+范数)
(normed space)　(特量,距离的特殊情况)

$x, y \in X$

$d: X \times X \mapsto R$

$(x, y) \to d(x, y)$

(1) 正定性

(2) 对称性

(3) $d(x,y) \le d(x,z) + d(y,z)$ 三角不等

* Hilbert space

范数: $X \mapsto R$

$x \mapsto \|x\|$

$x, y \in X$　$\|x-y\|$ 有意义

$x-y \in X$

$\|\cdot\| : x \mapsto (x, x)^{\frac{1}{2}}$

$[0,1]$ 上的连续函数全体 $C([0,1])$ 是一个向量空间

$\forall f, g \in C([0,1])$

$(f+g)(x) = f(x) + g(x)$

$\|f-g\| = \sup_{x\in[0,1]} |f(x)|$ 最大值

{实数原理}

确界原理

数集S

$k_1 < S < k_2$

则一定存在上确界
下确界

A的数值域半径.

$W(A) = \sup_{\|x\|=1} |\langle Ax, x \rangle|$

内积诱导的范数

线性算子: $f: X \to Y$

$f(x+y) = f(x) + f(y)$

$f(ax) = af(x)$

$(x, y) \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ 乘出的是二次函数

例: 在二维空间中的线性运算均可归为二阶矩阵

$r(A) \le w(A) \le \|A\|$

最远的sigma

$r(A) = \max\{ |\lambda| : \lambda \in \sigma(A) \}$

~~对于有限个数~~ sup=max

$\sigma(A) = \{ \lambda : |A - \lambda I| = 0$

$(A - \lambda I) v = 0$

非零.

$\begin{pmatrix} u & v \\ s & t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$a_1 a_2 = 0$

$a_1 a_1 = 4$

矩阵的范数

$\|Ax\| \le M\|x\|$ 　　M为常数

使上式成立的最小 M 称为 A 的范数，记为 $\|A\|$

2维空间内积定义

$$\left\langle \begin{pmatrix} x_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle = x_1\bar{x}_2 + y_1\bar{y}_2$$

$$|\langle x,y\rangle|^2 \le \langle x,x\rangle\langle y,y\rangle$$

$$\left|\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle\right|^2 = |x_1\bar{y}_1 + x_2\bar{y}_2|^2$$

$$\le \left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle = (|x_1|^2+|x_2|^2)(|y_1|^2+|y_2|^2)$$

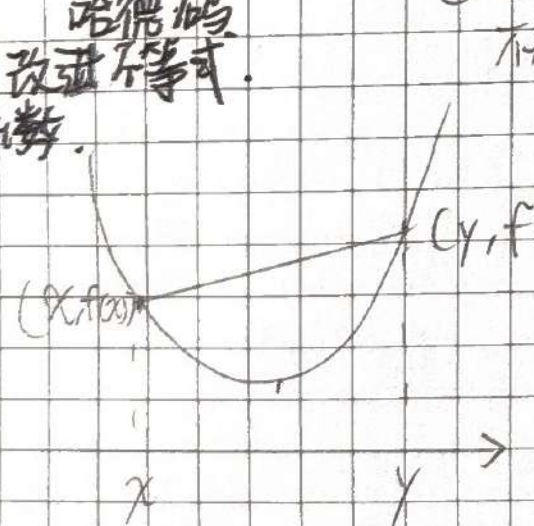
$$|x_1\bar{y}_1 + x_2\bar{y}_2| \le \sqrt{(|x_1|^2+|x_2|^2)(|y_1|^2+|y_2|^2)}$$

凸　y

不是凸函数

哈德码
敌越不等式.

凸函数.



凸函数图形：线段 ≤ 函数图形

$$f(\alpha x+(1-\alpha)y) \le \alpha f(x)+(1-\alpha)f(y)$$

严格凸函数：≤ 改为 <

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$f: H \to \mathbb{C}$

作用于 y，一定能写成

$f(y) = \langle x,y\rangle$ （x固定）

实数轴上的函数一定是 $y=kx$
$= \langle k,x\rangle$
里斯表示

T:
$\langle Tx,y\rangle = \langle x,T^*y\rangle$

$\mathrm{Re}\,T = \dfrac{T+T^*}{2}$ 　　$\mathrm{Im}\,T = \dfrac{T-T^*}{2i}$ 　　①参考文献1.

加权后：$R_v(T) = vT+(1-v)T^*$

$\mathrm{Im}_v(T) = \dfrac{vT-(1-v)T^*}{i}$ 　　$2t-2+2t$

$$w(T) = \sup_{\theta\in\mathbb{R}} \|\mathrm{Re}(e^{i\theta}T)\|$$ 　　$2t-2-2t$

$2(t-1)\cdot t$

参考文献1中

$$W_v(T) = \sup_{\theta\in\mathbb{R}} \|R_v(e^{i\theta}T)\|$$ 　　$4t^2-8t+4$

$$= \sup_{\theta\in\mathbb{R}} \|ve^{i\theta}T+(1-v)e^{-i\theta}T^*\|$$

$= ①$.

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$$w(T) = w[(1-2v)T^*+T]$$

$$= \sup_{\theta\in\mathbb{R}} \left\|\frac{e^{i\theta}[(1-2v)T^*+T]+e^{-i\theta}[(1-2v)T+T^*]}{2}\right\|$$

$$= \sup_{\theta\in\mathbb{R}} \left\|\frac{e^{i\theta}T+e^{-i\theta}T^*}{2} + \frac{i(2v-1)[e^{i\theta}T^*-e^{-i\theta}\ldots]}{2i}\right\|$$

$$= \sup_{\theta\in\mathbb{R}} \|\mathrm{Re}(e^{i\theta}T)-(2v-1)\mathrm{Re}(e^{-i\theta}T)\|$$

$$①:\ \sup_{\theta\in\mathbb{R}} \left\|\frac{e^{i\theta}T+e^{-i\theta}T^*}{2}+(2v-1)i\cdot\frac{e^{i\theta}T-e^{-i\theta}T^*}{2i}\right\|$$

$$= \sup_{\theta\in\mathbb{R}} \|\mathrm{Re}(e^{i\theta}T)+(2v-1)i\,\mathrm{Im}(e^{i\theta}T)\|$$

考虑：① $W_v(T+S) = W_v(T)+W_v(S)$

② $W_v^2(T+S) = W^2(T)+W^2(S)$

③ $\|T+S\|_v = \|T\|_v+\|S\|_v$

④ $\|T+S\|_v^2 = \|T\|_v^2+\|S\|_v^2$

计算
合理性：
参考文献 1

$\|T\|_v = \|vT + (1-v)T^*\|$

$\|T\|_0 = \|T^*\| = \|T\|$

$\|T\|_1 = \|T\|$

来源于参考文献 1：$w(T) = \sup_{\theta\in R} \left\| \frac{e^{i\theta}T + e^{-i\theta}T^*}{2} \right\|$

$= \sup_{\theta\in R} \|Re(e^{i\theta}T)\|$

考虑加权的是否相等 $w(T)$ $w_{\frac{1}{2}}(T)$

$w(T) = \sup_{\theta\in R} \|Re(e^{i\theta}T)\|$ $w_{\frac{1}{2}}(T) = \sup_{\theta\in R} \|Re(e^{i\theta}T)\|_{\frac{1}{2}}$

$\because (Re\,T)^* = (\frac{T+T^*}{2})^* = (\frac{T^*+T}{2}) = Re\,T$

$\therefore \|Re(e^{i\theta}T)\|_{\frac{1}{2}} = \|\frac{1}{2}Re(e^{i\theta}T) + \frac{1}{2}Re(e^{i\theta}T)\|$

$= \|Re(e^{i\theta}T)\|$

① $\|T+S\|_v = \|v(T+S) + (1-v)(T+S)^*\|$

$= \|vT + (1-v)T^*\| + \|vS + (1-v)S^*\|$

$= \|T\|_v + \|S\|_v$

② $\|\alpha T\|_v = \|v\alpha T + (1-v)\alpha T^*\|$

$= \alpha\, vT + (1-v)$

③ $\|\alpha A\|_t = \|(1-2t)\alpha^* A^* + \alpha A\|$

$= \|(1-2t)|\alpha| e^{-i\theta} A^* + |\alpha| e^{i\theta} A\|$

$= |\alpha| \|(1-2t)e^{i\theta} A + e^{-i\theta} A^*\|$

$\neq |\alpha| \|(1-2t)A^* + A\|$, $\forall A, t$

$\|\cdot\|_t$ 是范数，令 α 为一个常数

则在 $\forall \alpha \in C$, $A \in B(H)$ 的条件下有　　A为矩阵

$\|\alpha A\|_t = |\alpha| \|A\|_t$

$\|(1-2t)\alpha^* A^* + \alpha A\| = |\alpha| \|(1-2t)A^* + A\|$

$\|(1-2t)|\alpha| e^{-i\theta} A^* + |\alpha| e^{i\theta} A\| = |\alpha| \|(1-2t)A^* + A\|$

$\|(1-2t)e^{-i\theta} A^* + e^{i\theta} A\| = \|(1-2t)A^* + A\|$

即 $\left\| \frac{e^{-i\theta}A^* + e^{i\theta}A}{2} \right\| = \left\| \frac{A^*+A}{2} \right\|$

$\omega(A) = \frac{1}{2}\|A + A^*\|$

取 $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$，$t=0$

$\|A + A^*\| = \left\| \begin{matrix} 2 & \\ & -2 \end{matrix} \right\| = 4$

$\omega(A) = 2$

但按照

$r(A) \le \omega(A) \le \|A\|$

$2 \le 0$

不成立

Error: $\|\cdot\|_t$ is a norm. α ~~is~~ a constant

then $\forall \alpha \in \mathbb{C}$, $A \in B(H)$

satisfy $\|\alpha A\|_t = |\alpha| \|A\|_t$

$\exists \|(1-2t)\bar{\alpha} A^* + \alpha A\| = |\alpha| \|(1-2t)A^* + A\|$

$\|(1-2t)|\alpha| e^{-i\theta} A^* + |\alpha| e^{i\theta} A\| = |\alpha| \|(1-2t)A^* + A\|$

$\therefore \|(1-2t)e^{-i\theta} A^* + e^{i\theta} A\| = \|(1-2t)A^* + A\|$

即 $\left\|\frac{e^{-i\theta}A^* + e^{i\theta}A}{2}\right\| = \left\|\frac{A^* + A}{2}\right\|$

α

$\omega(A) = \frac{1}{2}\|A + A^*\|$

$w(A) = 2$

取 $A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$

$\frac{dy}{dx} > x^2$

$A + A^* = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ $\quad (22-x)(67-x) - \left(\frac{75}{2} + \frac{5}{3}\sqrt{3}i\right)\left(\frac{75}{2} - \frac{5}{3}\sqrt{3}i\right)$

$= \begin{pmatrix} 2 & \\ & -2 \end{pmatrix}$

$A x$

$\frac{dy}{dx}$

$2 \frac{dy}{dx}$

$r(A + A^*) = 2$

$\|A + A^*\| = 2$

$2y, x = Ax$

$(22-x)(67-x)$

$(97-x)(377-x) - 188 \times 188 = 0$

① $\omega_t(t)$ 感兴趣

$x \sin x$

$e^{i\theta} = \cos\theta + i\sin\theta$ $\quad -\cos\theta +$

$e^{i\theta} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$ $\quad e^{-i\theta} = \cos\theta - i\sin\theta$

$x \frac{dy}{dx} \sin x$

$= \frac{\sqrt{3}}{2} - \frac{1}{2}i$

$-\frac{5\sqrt{3}}{2}i$

$\|e^{i\theta}T + (1-2t)e^{-i\theta}T^*\|$

$= \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)\begin{pmatrix} 3 & 6 \\ 9 & 12 \end{pmatrix} + \left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

$\begin{pmatrix} \frac{3\sqrt{3}}{2} + \frac{3}{2}i & 3\sqrt{3} + 3i \\ \frac{9\sqrt{3}}{2} + \frac{9}{2}i & 6\sqrt{3} + 6i \end{pmatrix} + \begin{pmatrix} \frac{\sqrt{3}}{2} - \frac{1}{2}i & \frac{3\sqrt{3}}{2} - \frac{3}{2}i \\ \sqrt{3} - i & 2\sqrt{3} - 2i \end{pmatrix}$

$A = \begin{pmatrix} 2\sqrt{3} + i & \frac{9\sqrt{3}}{2} + \frac{3}{2}i \\ \frac{11\sqrt{3}}{2} + \frac{7}{2}i & 8\sqrt{3} + 4i \end{pmatrix}$ $\quad A^* = \begin{pmatrix} 2\sqrt{3} + \end{pmatrix}$

$\begin{pmatrix} 116 & \frac{349}{2} + \frac{15}{2}\sqrt{3}i \\ \frac{349}{2} + \frac{15}{2}\sqrt{3}i & 271 \end{pmatrix}$

$T + (1-2t)T^*$

$\begin{pmatrix} 3 & 6 \\ 9 & 12 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

$\|(1-2t)T^* + T\|$

正齐次性：

$\because (1-2t)T^* + T$ 为一个矩阵，

$\therefore (1-2t)\alpha T^* + \alpha T = \alpha[(1-2t)T^* + tT]$

$$e^{-i\theta} = (\frac{\sqrt3}{2} - \frac{1}{2}i) \quad e^{i\theta} = \frac{\sqrt3}{2} + \frac{1}{2}i$$

令 $f(t) = \|(1-2t)e^{-i\theta}T^* + te^{i\theta}T\|$，$g(t) = \|tT + (1-2t)T^*\|$

当 $\|\cdot\|$ 为范数时，$f(t) = g(t)$，即 $(f-g)(t) = 0$

$\theta = 30^\circ$，$T = \begin{pmatrix} 3 & 6 \\ 9 & 12 \end{pmatrix}$

$$f(t) = \|t(\frac{\sqrt3}{2} + \frac{1}{2}i)\begin{pmatrix} 3 & 6 \\ 9 & 12 \end{pmatrix} + (1-2t)(\frac{\sqrt3}{2} - \frac{1}{2}i)\begin{pmatrix} 3 & 9 \\ 6 & 12 \end{pmatrix}$$

$$\begin{pmatrix} \frac{3\sqrt3}{2}t + \frac{3}{2}it & 3\sqrt3 t + 3it \\ \frac{9\sqrt3}{2}t + \frac{9}{2}it & 6\sqrt3 t + 6it \end{pmatrix} + \begin{pmatrix} \frac{3}{2}\sqrt3 - \frac{3}{2}i & \frac{9}{2}\sqrt3 - \frac{9}{2}i \\ 3\sqrt3 - 3i & 6\sqrt3 - 6i \end{pmatrix}$$

$$- \begin{pmatrix} 3\sqrt3 t - 3it & 9\sqrt3 t - 9it \\ 6\sqrt3 t - 6it & 12\sqrt3 t - 12it \end{pmatrix}$$

$$g(t) = \|\begin{pmatrix} 3t & 6t \\ 9t & 12t \end{pmatrix} + \begin{pmatrix} 3 & 9 \\ 6 & 12 \end{pmatrix} - \begin{pmatrix} 6t & 18t \\ 12t & 24t \end{pmatrix}\|$$

$$f(t) = \left\|\begin{pmatrix} t(-\frac{3\sqrt3}{2} + \frac{9}{2}i) + \frac{3\sqrt3}{2} - \frac{3}{2}i & t(-6\sqrt3 + 6i) + \frac{9}{2}\sqrt3 - \frac{9}{2}i \\ t(-\frac{3}{2}\sqrt3 + \frac{21}{2}i) + 3\sqrt3 - 3i & t(-6\sqrt3 + 18i) + 6\sqrt3 - 6i \end{pmatrix}\right\|$$

$$\left\|\begin{pmatrix} -3t+3 & -12t+9 \\ -3t+6 & -12t+12 \end{pmatrix}\right\|$$

$f((1-t)A + tB)$

$f\begin{pmatrix} 2(1-t) & 0 \\ 2t & 0 \end{pmatrix}$

$$\left\|\begin{pmatrix} \frac{2(1-t)}{3} & 0 \\ \frac{2}{3}t & 0 \end{pmatrix} + \begin{pmatrix} \frac{4(1-t)}{3} & \frac{4t}{3} \\ 0 & \frac{2}{3}t \end{pmatrix}\right\|$$

$$= \left\|\begin{pmatrix} 2(1-t) & \frac{4t}{3} \\ \frac{2t}{3} & 0 \end{pmatrix}\right\|$$

$$A^*A = \begin{pmatrix} \frac{40}{9}t^2 - 8t + 4 & \frac{8t - 8t^2}{3} \\ \frac{8t - 8t^2}{3} & \frac{16t^2}{9} \end{pmatrix}$$

$$\lambda = \frac{4(7+3\sqrt5)(t-1)^2}{9}$$

$\left\|\begin{matrix} \frac{1}{3} & \frac{3}{2} \\ \frac{3}{2} & 0 \end{matrix}\right\|$

$\lambda = 1$

$\frac{1}{4}$ 0.25

$\lambda = 2.0811$

$\sqrt\lambda = 1.4426$

$\frac{1}{\sqrt\lambda}$ 0.7213

$\lambda = 2$

$\frac{1}{4\sqrt2} = 0.35$

1.3213

$f(A) + f(B)$

$\frac{\sqrt2 + 1}{2}$

$= 1.2071$

$$\int_0^1 \frac{2\sqrt{7+3\sqrt5}}{3}(1-t)\,dt = \frac{2\sqrt{7+3\sqrt5}}{3}\int_0^1 (1-t)\,dt = \frac{1}{2}\cdot\frac{\sqrt{7+3\sqrt5}}{3}$$

$\forall t \in (0,\frac{1}{2}) \cup (\frac{1}{2}, 0)$，都能找到 θ, T，使得

$\|e^{i\theta}T\|_t \neq \|T\|_t$

$\|(1-2t)e^{-i\theta}T^* + e^{i\theta}T\| \neq \|(1-2t)T^* + T\|$

用 $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ 对 $a\ b\ c\ d$ 进行调整，如令 $A<0$

三角不等式

$f_v(T)$

$\|A+B\| \leq \sim \leq \|A\| + \|A\|$

不是范数，但是有三角不等式

$\|A+B\|_t \leq$ 哈德码 $\leq \|A\|_t + \|B\|_t$

$f_t(A+B) \leq \sim \leq f_t(A) + f_t(B)$

$\|(1-2t)(A^*+B^*) + A + B\| \leq \|(1-2t)A^* + A\| + \|(1-2t)B^* + B\|$

用哈德玛先证凸函数 $g(v) = \|vA-(1-v)B\|_t$

$g(v) = \| t[v(A)+(1-v)B] + (1-t)[v(A)+(1-v)B]^* \|$

联想可能是标量性质

$F_v(T) = vT + (1-v)T^*$

$f_v(T) = \|vT + (1-v)T^*\|$

$f_v(A+B) \leq f_v(A) + f_v(B)$

$T = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ $S\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$

$\lambda = \frac{1}{2}$ $v = \frac{1}{3}$

哈德码过程

$f(T) = \|vT + (1-v)T^*\|$

$f(t) = \|tT + (1-t)S\|_v = \|v(tT+(1-t)S) + (1-v)(tT^* + (1-t)S^*)\|$

$\|v(\frac{1}{2}T + \frac{1}{2}S) + (1-v)(\frac{1}{2}T^* + \frac{1}{2}S^*)\|$

$= \|\frac{1}{3}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\|$

$= \|\begin{pmatrix} 1 & \frac{2}{3} \\ \frac{1}{3} & 0 \end{pmatrix}\|$

$= \begin{pmatrix} 1 & \frac{1}{3} \\ \frac{2}{3} & 0 \end{pmatrix}\begin{pmatrix} 1 & \frac{2}{3} \\ \frac{1}{3} & 0 \end{pmatrix}$

$= \begin{pmatrix} \frac{10}{9} & \frac{2}{3} \\ \frac{2}{3} & \frac{4}{9} \end{pmatrix}$

$\lambda^2 - \frac{14}{9}\lambda + \frac{4}{81} = 0$

$\lambda_{max} = \frac{-(-\frac{14}{9}) + \sqrt{(-\frac{14}{9})^2 - 4 \cdot \frac{4}{81}}}{2}$

$= 1.5231 = \frac{3\sqrt{5}+7}{9}$ 0.6720

$f(\frac{A+B}{2}) = 1.2342$

$\frac{1}{2}f\left(\frac{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}}{2}\right) + \frac{1}{2}f\left(\frac{\begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}{2}\right)$

$= \frac{1}{2}f\begin{pmatrix} \frac{1}{2} & 0 \\ \frac{3}{2} & 0 \end{pmatrix} + \frac{1}{2}f\begin{pmatrix} \frac{3}{2} & 0 \\ \frac{1}{2} & 0 \end{pmatrix}$

$\begin{pmatrix} \frac{1}{6} & 0 \\ \frac{1}{2} & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{3} & 1 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{6} & 0 \end{pmatrix} + \begin{pmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{pmatrix}$

$= \frac{1}{2}\|\begin{pmatrix} \frac{5}{6} & 1 \\ \frac{1}{2} & 0 \end{pmatrix}\| + \frac{1}{2}\|\begin{pmatrix} \frac{3}{2} & \frac{1}{3} \\ \frac{1}{6} & 0 \end{pmatrix}\|$

$\|\begin{pmatrix} \frac{5}{6} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}\| + \|\begin{pmatrix} \frac{3}{2} & \frac{1}{6} \\ \frac{1}{3} & 0 \end{pmatrix}\|$

$\begin{pmatrix} \frac{34}{36} & \frac{5}{6} \\ \frac{5}{6} & 1 \end{pmatrix}$ $\begin{pmatrix} \frac{82}{36} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{9} \end{pmatrix}$

$\lambda^2 - \frac{35}{18}\lambda + \frac{1}{4} = 0$

$\lambda_{max} = \frac{35 + \sqrt{901}}{36} = 1.8060$

$\sqrt{\lambda} = 1.3439$

$\lambda^2 - \frac{43}{18}\lambda + \frac{1}{324} = 0$

$\lambda_{max} = \frac{3\sqrt{205} + 43}{36}$

$= 2.3876$

$\sqrt{\lambda} = 1.5452$

$\frac{16}{}$ 1.7226 … 1.4446

三角不等式什么时候取等

$\|T+S\| = \|T\| + \|S\|$

$\Rightarrow \|v(T+S) + (1-v)(T+S)^*\| = \|vT + (1-v)T^*\| + \|vS + (1-v)S^*\|$

$f_v(T+S) = f_v(T) + f_v(S)$

Thm. 2.4 新的证明

i. $\|T+S\| = \|T\| + \|S\|$

ii. $\exists \{x_n\}$ s.t. $\lim\limits_{n\to\infty} \langle Tx_n, Sx_n\rangle = \|T\|\|S\|$ "可列集"

i → ii: $\exists$ 一列点 $\{x_n\}$ 使得 $\|(T+S)x_n\| = \|T\| + \|S\|$

$\Rightarrow \langle (T+S)x_n + (T+S)x_n\rangle^{\frac{1}{2}} = \|T\| + \|S\|$

$\Rightarrow \langle (T+S)x_n, (T+S)x_n\rangle = (\|T\| + \|S\|)^2$

$\langle Tx_n, Tx_n\rangle + \langle Tx_n, Sx_n\rangle + \langle Sx_n, Tx_n\rangle + \langle Sx_n, Sx_n\rangle = (\|T\|+\|S\|)^2$

① $\forall \{x_n\}$, $\lim\limits_{n\to\infty} \langle Tx_n, Sx_n\rangle$ 极限不存在

② $\forall \{x_n\}$, $\lim\limits_{n\to\infty} \langle Tx_n, Sx_n\rangle = \|T\|\|S\| - \varepsilon$ 是实数

若 $\lim\limits_{n\to\infty} \langle Tx_n, Sx_n\rangle > \|T\|\|S\|$

$\lim\limits_{n\to\infty} a_n A > C$

$\forall \varepsilon\ \exists N,\ |a_n - A| < \varepsilon$

$-\varepsilon < A_n - A < \varepsilon$

$\varepsilon = A - C > 0$

$A - \varepsilon < A_n < A + \varepsilon$

$C < A_n$

$\therefore n > N$ 时 $\langle Tx_n, Sx_n\rangle > \|T\|\|S\|$

不符合 Caughchy

对于②

②

$\langle Tx_n, Tx_n\rangle \le \|Tx_n\|\|Tx_n\|$ Cauchy

$\le \|T\|\|x_n\|\|T\|\|x_n\|$ 范数定义

$\le \|T\|^2$ 模长为1

$\therefore \le \|T\|^2 + \|S\|^2 + 2\langle Tx_n, Sx_n\rangle$

$(\|T\|+\|S\|)^2 \le \|T\|^2 + \|S\|^2 + 2(\|T\|\|S\| - \varepsilon)$

不符合② $\therefore =$

令 $T = \begin{bmatrix} 2a & 2b \\ 2c & 2d \end{bmatrix}$ $\quad e^{i\theta} = \frac{\sqrt{3}}{2} + \frac{1}{2}i \quad e^{-i\theta} = \frac{\sqrt{3}}{2} - \frac{1}{2}i \quad t = \frac{1}{3}$

$\|(1-2t)e^{-i\theta}T^* + te^{i\theta}T\| \neq \|(1-2t)T^* + T\|$

$\|A\| = (e^{-i\theta}T^* - 2te^{-i\theta}T^* + e^{i\theta}T \quad 2\sqrt{3}t - 2it)$

$= \begin{pmatrix} (\sqrt{3}-i)a & (\sqrt{3}-i)c \\ (\sqrt{3}-i)b & (\sqrt{3}-i)d \end{pmatrix} - \begin{pmatrix} 2t(\sqrt{3}-i)a & 2t(\sqrt{3}-i)c \\ 2t(\sqrt{3}-i)b & 2t(\sqrt{3}-i)d \end{pmatrix} + \begin{pmatrix} (\sqrt{3}+i)a & (\sqrt{3}+i)b \\ (\sqrt{3}+i)c & (\sqrt{3}+i)d \end{pmatrix}$

$A = \begin{pmatrix} (2\sqrt{3} - 2\sqrt{3}t + 2it)a & (1-2t)(\sqrt{3}-i)c + (\sqrt{3}+i)b \\ (1-2t)(\sqrt{3}-i)b + (\sqrt{3}+i)c & (2\sqrt{3} - 2\sqrt{3}t + 2it)d \end{pmatrix}$

$\lambda_{A^*A}^{1/2}$ $\quad A^*A = \begin{pmatrix} a_1 & c_1 + d_1 \\ c_1 - d_1 & b_1 \end{pmatrix}$

$B^*B = \begin{pmatrix} a_2 & c_2 + d_2 i \\ c_2 - d_2 i & b_2 \end{pmatrix} \quad c_2^2 + d_2^2 \quad B = \begin{pmatrix} 4(1-t)a & 2(1-2t)(b+c) \\ 2(1-2t)(b+c) & 4(1-t)d \end{pmatrix}$

$((a_1 - \lambda)(b_1 - \lambda)$

$d_1 \quad \lambda^2 - (a_1 + b_1)\lambda + a_1 b_1 +$

$\lambda^2 - (2\sqrt{3} - 2\sqrt{3}t + 2it)(a+d)\lambda + (2\sqrt{3} - 2\sqrt{3}t + 2it)^2 ad - (1-2t)^2(3-i)^2 bc - 4(1-2t)(b^2 + c^2)$

$-(2 + 2i\sqrt{3})bc = 0$

$\begin{pmatrix} (1-2t)2a & (1-2t)2c \\ (1-2t)2b & (1-2t)2d \end{pmatrix} + \begin{pmatrix} 2a & 2b \\ 2c & 2d \end{pmatrix}$

$\lambda_{B\,max} = \frac{a_2 + b_2 + \sqrt{(a_2 - b_2)^2 - 4(c_2^2 + d_2^2)}}{}$

$\begin{pmatrix} e^{-i\theta}(1-2t)a & c \\ b & d \end{pmatrix} + \begin{pmatrix} e^{i\theta}a & b \\ c & d \end{pmatrix}$

$[2\cos\theta - 2t(\cos\theta - i\sin\theta)]a \quad (\cos\theta - i\sin\theta)(1-2t)c + (\cos\theta + i\sin\theta)b$

$(\cos\theta - i\sin\theta)(1-2t)b + (\cos\theta + i\sin\theta)c \quad [2\cos\theta - 2t(\cos\theta - i\sin\theta)]d$

$[2\cos\theta - 2t(\cos\theta + i\sin\theta)]a$

$(\cos\theta + i\sin\theta)(1-2t)c + (\cos\theta - i\sin\theta)b$

$A = \begin{pmatrix} a(e^{-i\theta}(1-2t)) + e^{i\theta}) & c(e^{-i\theta}(1-2t)) + be^{i\theta} \\ b(e^{-i\theta}(1-2t)) + ce^{i\theta} & d(e^{-i\theta}(1-2t) + e^{i\theta}) \end{pmatrix} \quad ((1-2t)e^{-i\theta})c$

$A^* = \begin{pmatrix} a(e^{i\theta}(1-2t)) + e^{-i\theta}) & b(e^{i\theta}(1-2t)) + ce^{-i\theta} \\ c(e^{i\theta}(1-2t)) + be^{-i\theta} & d(e^{i\theta}(1-2t) + e^{-i\theta}) \end{pmatrix}$

$A^*A = ? \quad a^2((1-2t)^2 + e^{2i\theta}(1-2t) + e^{-2i\theta}(1-2t) + 1) +$

$b^2 + bce^{2i\theta}(1-2t) + bce^{-2i\theta}(1-2t) + c^2$

$a^2(1-2t)^2 + e^{2i\theta}(1-2t) + e^{-2i\theta}(1-2t) + 1 + b^2 + c^2 + b\,c\,e^{2i\theta}(1-2t) + bce^{-2i\theta}(1-2t)$

$ac(1-2t)^2 + ab\,e^{2i\theta}(1-2t) + ce^{-2i\theta}(1-2t) + b + bd + c + b(\cdot) + cd(\cdot)$

$bc(1-2t)^2 + ce^{2i\theta}(1-2t) + abe^{-2i\theta}(1-2t) + b + bd + c + cd(+) + b(-)$

$a^2 + b^2 + c^2 + bc(+) + bc(-) + d^2 + 1 + d(+) + d(-)$

$(1-2t)^2$

$a=0,\ d=1$

$b^2+c^2+ \quad bc\,2\cos 2\theta(1-2t)+1 \quad [(c+b)\cos 2\theta - (c-b)\sin 2\theta](1-2t) + b + c$

$[(c+b)\cos 2\theta + (c-b)\cos 2\theta](1-2t) + b + c \quad b^2+c^2(1-2t)^2 + bc\,2\cos 2\theta(1-2t)+1$

$c(\cos 2\theta + i\sin 2\theta)(1-2t) + b(\cos 2\theta - i\sin 2\theta)(1-2t)$

$\|(1-2t)T^* + T\| \qquad T^* = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \quad T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$= \begin{pmatrix} (1-2t)a & (1-2t)c \\ (1-2t)b & (1-2t)d \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$B = \begin{pmatrix} 2(1-t)a & (1-2t)c+b \\ (1-2t)b+c & 2(1-t)d \end{pmatrix}$

$B^* = \begin{pmatrix} 2(1-t)a & (1-2t)b+c \\ (1-2t)c+b & 2(1-t)d \end{pmatrix}$

$\cancel{4(1-t)^2a^2} + (1-2t)b^2 + 2(1-2t)bc + c^2$

$2(1-t)(1-2t)ac + 2(1-t)ab + 2(1-t)d(1-2t)b + 2(1-t)dc$

$2(1-t)(1-2t)ac + 2(1-t)ab + 2(1-t)(1-2t)bd + 2(1-t)dc$

$(1-2t)c^2 + 2(1-2t)bc + c^2 + 4(1-t)^2 d^2 \quad 0$

$((1-2t)b+c)^2 - \lambda \quad 0 \quad 2(1-t)dc \quad d$

0

$((1-2t)b+c)^2 - \lambda$

$((1-2t)b+c)^4 - 2\lambda((1-2t)b+c)^2 + \lambda^2 = 0$

$\lambda = \dfrac{2((1-2t)b+c)+ \quad 4}{2}$

$= (1-2t)b+c$

$\|(1-2t)e^{-i\theta}T^*+e^{i\theta}T\| \neq \|(1-2t)T^*+e^{i\theta}T\|$ 令 $T=\begin{pmatrix}a&b\\c&d\end{pmatrix}$

$$\begin{pmatrix}(1-2t)e^{-i\theta}a & (1-2t)e^{-i\theta}c\\ (1-2t)e^{-i\theta}b & (1-2t)e^{-i\theta}d\end{pmatrix}+\begin{pmatrix}e^{i\theta}a & e^{i\theta}b\\ e^{i\theta}c & e^{i\theta}d\end{pmatrix}$$

$$=\begin{pmatrix}[(1-2t)e^{-i\theta}+1]a & (1-2t)e^{-i\theta}c+e^{i\theta}b\\ (1-2t)e^{-i\theta}b+e^{i\theta}c & [(1-2t)e^{-i\theta}+1]d\end{pmatrix} \quad A^*=\begin{pmatrix}[(1-2t)e^{i\theta}+1]a & (1-2t)e^{i\theta}b+e^{-i\theta}c\\ (1-2t)e^{i\theta}c+e^{-i\theta}b & [(1-2t)e^{i\theta}+1]d\end{pmatrix}$$

$$=\begin{pmatrix}a^2[(1-2t)^2+(1-2t)(e^{i\theta}+e^{-i\theta})+1]+(1-2t)^2b^2+(1-2t)e^{2i\theta}bc+(1-2t)e^{-2i\theta}bc+c^2\\ (1-2t)^2ac+(1-2t)e^{2i\theta}ab+(1-2t)e^{-i\theta}ac+e^{i\theta}ab+(1-2t)^2bd+(1-2t)e^{-2i\theta}cd+(1-2t)e^{i\theta}bd+e^{-i\theta}cd\\ (1-2t)^2ac+(1-2t)e^{-2i\theta}ab+(1-2t)e^{i\theta}ac+e^{-i\theta}ab+(1-2t)^2bd+(1-2t)e^{2i\theta}cd+(1-2t)e^{-i\theta}bd+e^{i\theta}cd\\ d^2[(1-2t)^2+(1-2t)(e^{i\theta}+e^{-i\theta})+1]+(1-2t)^2c^2+(1-2t)e^{2i\theta}bc+(1-2t)e^{-2i\theta}bc+b^2\end{pmatrix}$$

$\|(1-2t)T^*+T\|$ 令 $T=\begin{pmatrix}a&b\\c&d\end{pmatrix}$

$$\begin{pmatrix}(1-2t)a & (1-2t)c\\ (1-2t)b & (1-2t)d\end{pmatrix}+\begin{pmatrix}a&b\\c&d\end{pmatrix}$$

$$=\begin{pmatrix}(1-2t)a+a & (1-2t)c+b\\ (1-2t)b+c & (1+2t)d+d\end{pmatrix} \quad B^*=\begin{pmatrix}(1-2t)a+a & (1-2t)b+c\\ (1-2t)c+b & (1-2t)d+d\end{pmatrix}$$

$$=\begin{pmatrix}(1-2t)^2a^2+2(1-2t)a^2+a^2+(1-2t)^2b^2+2(1-2t)bc+c^2\\ (1-2t)^2ac+(1-2t)(ab+ac)+ab+(1-2t)^2bd+(1-2t)(cd+bd)+cd\\ (1-2t)^2ac+(1-2t)(ab+ac)+ab+(1-2t)^2bd+(1-2t)(cd+bd)+cd\\ (1-2t)^2d^2+2(1-2t)d^2+d^2+(1-2t)c^2+2(1-2t)bc+b^2\end{pmatrix}$$

$$\begin{pmatrix}(1-2t)^2+(1-2t)(e^{i\theta}+e^{-i\theta})+1+1^2 & (1-2t)^2+(1-2t)e^{-i\theta}+(1-2t)e^{-2i\theta}+e^{-i\theta}\\ (1-2t)^2+(1-2t)e^{i\theta}+(1-2t)e^{2i\theta}+e^{i\theta} & (1-2t)^2+(1-2t)(e^{i\theta}+e^{-i\theta})+1+(1-2t)^2\end{pmatrix}$$

$$\begin{pmatrix}(1-2t)^2+2(1-2t)+1+1^2 & (1-2t)^2+2(1-2t)+1\\ (1-2t)^2+2(1-2t)+1 & (1-2t)^2+2(1-2t)+1+(1-2t)^2\end{pmatrix}$$

证明单调性

若 $P_n^2(t)-q_m(t)$ 是单调，则 $P_n(t)+\sqrt{q_m(t)}=0$ 为单调

$x=2-2t$

$$T = u|T|,\quad T^{D} = |T|u,\quad T^{*} = |T|u^{*}$$

$$e^{i\theta} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$w_v(T) = \sup_{\theta\in\mathbb{R}} \| v e^{i\theta} T + (1-v) e^{-i\theta} T^{*} \|$$

$$T = \begin{pmatrix} 3 & 6 \\ 9 & 12 \end{pmatrix} \quad T^{*} = \begin{pmatrix} 3 & 9 \\ 6 & 12 \end{pmatrix}$$

$$\begin{pmatrix} \frac{3\sqrt{3}}{2} + \frac{3}{2}i & 3\sqrt{3} + 3i \\ \frac{9\sqrt{3}}{2} + \frac{9}{2}i & 6\sqrt{3} + 6i \end{pmatrix} + \begin{pmatrix} \frac{\sqrt{3}}{2} - \frac{1}{2}i & \frac{3\sqrt{3}}{2} - \frac{3}{2}i \\ \sqrt{3} - i & 2\sqrt{3} - 2i \end{pmatrix} = \begin{pmatrix} \sqrt{3} & \frac{5}{2}\sqrt{3} - \frac{1}{2}i \\ & 4\sqrt{3} \end{pmatrix}$$

$$\begin{pmatrix} 2\sqrt{3} + i & \frac{9}{2}\sqrt{3} + \frac{3}{2}i \\ \frac{11}{2}\sqrt{3} + \frac{7}{2}i & 8\sqrt{3} + 4i \end{pmatrix}$$

$$\| t e^{i\theta} T + (1-t) e^{-i\theta} T^{*} \| \qquad T = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

$$\left\| \begin{pmatrix} it & 0 \\ 0 & it \end{pmatrix} \right\| + \left\| \begin{pmatrix} ti - i & 0 \\ 0 & ti - i \end{pmatrix} \right\|$$

$$A = \left\| \begin{pmatrix} 2it - i & 0 \\ 0 & 2it - i \end{pmatrix} \right\| \qquad A^{*} = \begin{pmatrix} i - 2it & \\ & i - 2it \end{pmatrix}$$

$$A^{*}A = \begin{pmatrix} (i-2it)(2it-i) & \\ & (i-2it)(2it-i) \end{pmatrix}$$

$$= \begin{pmatrix} (2t-1)^2 & \\ & (2t-1)^2 \end{pmatrix}$$

$$\|A^{*}A\| = |2t-1|$$

$$\| t T + (1-t) T^{*} \| \qquad |2t-1| = 1$$

$$\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} + \begin{pmatrix} 1-t & 0 \\ 0 & 1-t \end{pmatrix} \qquad x = t_1,\ y = t_2$$

$$A^{*}A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

证明凸函数 $f(t) = \| v(tT + (1-t)S) + (1-v)(tT^{*} + (1-t)S^{*}) \|$

$$= \| v[(\alpha t_1 + \beta t_2) T + (1 - \alpha t_1 - \beta t_2) S] + (1-v)[(\alpha t_1 + \beta t_2) T^{*} + (1 - \alpha t_1 - \beta t_2) S^{*}] \|$$

$$= \| v(\alpha t_1 T + \beta t_2 T + S - \alpha t_1 S - \beta t_2 S) + (1-v)(\alpha t_1 T^{*} + \beta t_2 T^{*} + S^{*} - \alpha t_1 S^{*} - \beta t_2 S^{*})$$

$$= \| \alpha[v(t_1 T - t_1 S) + (1-v)(t_1 T^{*} - t_1 S^{*})] + \beta[v(t_2 T - t_2 S) + (1-v)(t_2 T^{*} - t_2 S^{*})] + vS + (1-v)S^{*}$$

$$(\alpha+\beta) vS + (\alpha+\beta)(1-v) S^{*}$$

$$= \| \alpha\{ v[t_1 T + (1-t_1) S] + (1-v)(t_1 T^{*} + (1-t_1) S^{*}) \} + \beta\{ v[t_2 T + (1-t_2) S] + (1-v)(t_2 T^{*} + (1-t_2) S^{*}) \} \|$$

$\leq \alpha \| v(t_1 T+(1-t_1)S)+(1-v)(t_1 T^*+(1-t_1)S^*)\| +$
$\beta \| v(t_2 T+(1-t_2)S)+(1-v)(t_2 T^*+(1-t_2)S^*)\|$

$\beta v S \quad \beta(1-v)S$

need: $\alpha v S \quad \alpha(1-v)S^*$
$\beta v S \quad \beta(1-v)S^*$

$-2t_1+4t^2$

$-(2_1 t-$

$\cancel{-4t^2}+4-8t+\cancel{4t^2}+\frac{1}{2}\sqrt{\quad}=0$

$16t-8$

$\sqrt{\quad}=\cancel{8t^2}\ \cancel{8-16t}$

命题. 基本性质
主要结果 定理
有性质. 引理
推论 有趣的事实

$F_v(T)=vT+(1-v)T^*$, $\widetilde{F_T}(v)=vT+(1-v)T^*$
$f_v(T)=\|vT+(1-v)T^*\|=\|F_v(T)\|$
$f_T(v)=\|vT+(1-v)T^*\|=\|F_v(T)\|$

Def. $\forall v\in[0,1]$, 如下定义的 $F_v(T)$
$F_v(T)=vT+(1-v)T^*$
称为T的 F_v 变换

Proposition 若 $T\in B(H)$ 则下述条件等价
(i) T是对称矩阵
(ii) $F_v(T)=T$ 对任意的 $v\in[0,1]$
(iii) $F_v(T)=T$ 对某一个 $v\in[0,1]$
(iv) $F_v(T)=T^*$ 对某一个 $v\in[0,1]$
(v) $F_v(T)=T^*$ 对任意的 $v\in[0,1]$

证明: 注意到 $F_v(T)=T$, $v\neq1$ 时 $\Leftrightarrow vT+(1-v)T^*=T$ $v\neq1$
$\Leftrightarrow T=T^*$

(i) $\Rightarrow$ (ii) 若 $T=T^*$, 则 $\forall v\in(0,1)$, $F_v(T)=vT+(1-v)T^*=vT+(1-v)T=T$

(ii) $\Rightarrow$ (iii) 显然成立

(ii) $\Rightarrow$ (iv) 因为 $\exists v\in(0,1)$ 成立 $F_v(T)=T$, 则 $vT+(1-v)T^*=T$
$\Rightarrow T^*=T$, 故 $F_v(T)=T=T^*$

(iv) ⇒ (v) 因为 $\exists v \in (0,1)$ 成立

$F_v(T) = T^*$，则 $vT + (1-v)T^* = T^*$

$\Rightarrow T = T^*$，故 $\forall v \in (0,1)$，有

$F_v(T) = vT + (1-v)T^* = vT^* + (1-v)T^* = T^*$

(v) ⇒ (i) 显然成立

Remark 简单地，可以看出 $v \in [0,1]$，对任意的算子 T，均有 $F_v(T)$ 开或 ...

命题. 若 $T \in B(H)$，$v \neq \frac{1}{2}$，则 $F_v(T) = 0$ 当且仅当 $T = 0$

证 若 $v = 1$ 时 $T = F_v(T) = 0$ 成立

若 $v \neq 1$ 时，则 $F_v(T) = vT + (1-v)T^* = 0$ (1)

$\Rightarrow T^* = \frac{v}{v-1} T$ (2)

等式(1)两边取 * 后有

$vT^* + (1-v)T = 0$ (3)

将(2)代入(3) $\left(\frac{v^2}{v-1} + (1-v)\right)T = 0$

又 $v \neq \frac{1}{2}$，故 $\frac{v^2}{v-1} + 1 - v \neq 0$，从而 $T = 0$

⇐ 显然

Remark 在 $v = \frac{1}{2}$ 时，只要 $T = -T^*$，则 T 是反对称算子

定理. 若 $T \in B(H)$，$v \in [0,1]$，则 $\|F_v(T)\| \leq \omega \leq \|T\|$

证明: $|\langle F_v(T)x, y\rangle| = |\langle (vT + (1-v)T^*)x, y\rangle|$

$= |\langle vTx, y\rangle + \langle (1-v)T^*x, y\rangle|$

$\leq |\langle vTx, y\rangle| + |\langle (1-v)T^*x, y\rangle|$

$= v|\langle Tx, y\rangle| + (1-v)|\langle T^*x, y\rangle|$

$\leq v\langle |T|x, x\rangle^{\frac{1}{2}}\langle |T^*|y, y\rangle^{\frac{1}{2}} + (1-v)\langle |T^*|x, x\rangle^{\frac{1}{2}}\langle |T|y, y\rangle^{\frac{1}{2}}$

$\leq v\frac{\langle |T|x, x\rangle + \langle |T^*|y, y\rangle}{2} + (1-v)\frac{\langle |T^*|x, x\rangle + \langle |T|y, y\rangle}{2}$

$= \frac{\langle (v|T| + (1-v)|T^*|)x, x\rangle + \langle (v|T^*| + (1-v)|T|)y, y\rangle}{2}$

$\leq \frac{\|v|T| + (1-v)|T^*|\| + \|v|T^*| + (1-v)|T|\|}{2}$

不等式两边关于 $\|x\| = \|y\| = 1$ 取上确界

$\sup_{\|x\| = \|y\| = 1} |\langle F_v(T)x, y\rangle| \leq \sim$

即 $\|F_v(T)\| \leq \sim \leq \|T\|$

可联合得到

预备知识
定理：混合Schwarz不等式
若 $T\in B(H)$，则
$$|\langle Tx,y\rangle|^2\le\langle |T|x,x\rangle\langle |T^*|y,y\rangle$$
其中 $|T|=(T^*T)^{\frac{1}{2}}$，$|T^*|=(TT^*)^{\frac{1}{2}}$

$$\sim\le\frac{v\||T|\|+(1-v)\||T^*|\|+v\||T^*|\|+(1-v)\||T|\|}{2}$$

$$=\frac{\||T|\|+\||T^*|\|}{2}$$

$\forall A\in B(H)$
$\||A|\|=\|(A^*A)^{\frac{1}{2}}\|=\|(A^*A)^{\frac{1}{2}}(A^*A)^{\frac{1}{2}}\|^{\frac{1}{2}}$
$=\|A^*A\|^{\frac{1}{2}}$

$$=\frac{\|T\|+\|T^*\|}{2}$$

$=\|T\|$

即证 $\|F_v(T)\|\le ○\le\|T\|$

所以 $f_T(v)=\|vT+(1-v)T^*\|$
$\le f_T(1)$
$=f_T(0)$

推论 若 $T\in B(H)$，则
$\omega_v(T)\le ○\le\|T\|$

$|e^{i\theta}T|$
$=(e^{-i\theta}T^*e^{i\theta}T)^{\frac{1}{2}}$
$=(T^*T)^{\frac{1}{2}}$
$=|T|$

证：由定理有
$\|F_v(e^{i\theta}T)\|\le ○\le\|e^{i\theta}T\|=\|T\|$
对 $\theta\in\mathbb{R}$ 取上确界
则 $\omega_v(T)\le ○\le\|T\|$

Remark 我们加强了参考文献1的结果。

$F_\nu(T+S)$
$= \|\nu(T+S)+(1-\nu)(T+S)^*\| \le c \le \|T\|_\nu + \|S\|_\nu$

$>$

$\oplus$ Let $T = e^{i\theta}T,\ S = e^{i\theta}S$

$W_\nu(T+S) = \|\nu(e^{i\theta}(T+S)) + (1-\nu)e^{-i\theta}(T+S)^*\| \le 0$

$\le \|\nu e^{i\theta}T + (1-\nu)e^{-i\theta}T^*\| + \|\nu e^{i\theta}S + (1-\nu)e^{-i\theta}S^*\|$

$\le \sup + \sup$

$= W_\nu(T) + W_\nu(S)$

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$\nu = \frac{1}{2} \qquad T = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \qquad S = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$

$f(t) = \|\nu(tT + (1-t)S) + (1-\nu)(tT^* + (1-t)S^*)\|$

$\left\| \begin{pmatrix} t & 0 \\ 1-t & 0 \end{pmatrix} + \begin{pmatrix} t & 1-t \\ 0 & 0 \end{pmatrix} \right\|$

$= 2\left\| \begin{pmatrix} t & 1-t \\ 0 & 1-t \end{pmatrix} \right\|$

$\frac{58}{4} \qquad \frac{29}{20}$

$A^*A = \begin{pmatrix} t^2 & 0 \\ 0 & (1-t)^2 \end{pmatrix}$

$((1-t)^2-\lambda)(4t^2+(1-t)^2-\lambda) - 4t^2(1-t)^2 = 0$

$\cancel{4t^2(1-t)^2} + (1-t)^4 - \lambda(1-t)^2 - 4\lambda t^2 - \lambda(1-t)^2 + \lambda^2 = 4t^2(1-t)^2$

$\lambda^2 - \lambda[2(1-t)^2 + 4t^2] + (1-t)^4 = 0$

$\sqrt{\lambda} = \sqrt{3t^2 - 2t + 1 + 2t\sqrt{2t^2 - 2t + 1}}$

$\frac{f(0)+f(1)}{2} = \frac{\|S\|_\nu + \|T\|_\nu}{2} = \frac{\|\frac{1}{2}S + \frac{1}{2}S^*\| + \|\frac{1}{2}T + \frac{1}{2}T^*\|}{2}$

$0.93 + \frac{2 - 2\lambda + \lambda}{2}$

$1+2 \qquad 1$

$\frac{\left\| \begin{pmatrix} 0 & 1 \\ 1 & d \end{pmatrix} \right\| + \left\| \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \right\|}{2} = 1.5$

$\frac{1}{2}[f(\lambda) + (1-\lambda)f(1) + \lambda f(0)]$

$0.62 \qquad 2 + \lambda + \sqrt{2}\lambda$

$f(\lambda) = \left\| \begin{pmatrix} \lambda & 0 \\ 1-\lambda & 0 \end{pmatrix} + \begin{pmatrix} \lambda & 1-\lambda \\ 0 & 0 \end{pmatrix} \right\|$

$= \left\| \begin{pmatrix} 2\lambda & 1-\lambda \\ 1-\lambda & 0 \end{pmatrix} \right\|$

$1.815 + 0.882$

$2 - \lambda \qquad 2 + (\sqrt{2}-1)\lambda$

$\frac{2+\lambda}{2}$

$2t^2-2t+1 > 0.9 \quad 0.5x$ $\quad \begin{pmatrix} 2(1+\lambda) \\ 2(1-\lambda) \end{pmatrix}$ $\quad 20t^2-19t+1>0$

$2t^2-\frac{3}{2}t+0.1>0 \quad 20t^2$ $\quad 20(t^2-\frac{3}{4}t+\frac{9}{64})$

$(\frac{1}{2}, \frac{1}{2})$

$0.9 \quad 0.825t$

$(1-2t)\sqrt{2t^2-2t+1}$

考虑 $t\in[0,\frac{1}{2}]$ $\sim < 1-\frac{1}{2}t$ $\quad \frac{1}{5}t+0.6 \sim$

说明 $t\in[\frac{1}{2},1]$ $\sim < \frac{1}{5}t+0.8$ $\quad \frac{1}{2}t+0.9 \le \sim$ $\quad 4x-2$

三阶积分

$t\in[\frac{1}{2},1]$, $\frac{1}{5}x+0.5 < 2t^2-2t+1 < \frac{1}{5}x+0.8$

$= \int_{\frac{1}{2}}^{1} \sim + \int_{0}^{\frac{1}{2}} \sim$

$\lambda(\lambda-2)=0$

任务：$M_v(T)$ 估计

$\sqrt{2t^2-2t+1} > \sqrt{t}$

$W_v(T+S)$

$2t^2-2t+1 \quad \frac{9}{10}t$

$\begin{matrix} 2 & 1 \\ 1 & 0 \end{matrix}$

$T=\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ $S=\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$

$5 \mid 825$, $5 \mid 165$, 33

$f(\frac{1}{2})$

$\sqrt{2t^2-2t+1} > \sqrt{t} \quad \sqrt{\frac{33}{40}}t$

$= \| \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \|$

$2t^2-3t+1$

$= \| \begin{matrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{matrix} \|$

$= 2(t^2-\frac{3}{2}t+\frac{9}{16})-\frac{1}{8}$

$2t^2-2t+1 = \frac{1}{4}t^2 + \frac{4}{5}t + \frac{16}{25}$ $\quad A^*A = \begin{pmatrix} \frac{5}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix}$

$0.8-$

$\frac{7}{4}t^2-$

$\sqrt{\lambda_{max}} = \sqrt{\frac{\sqrt{2}}{2}+\frac{3}{4}} = \sqrt{\frac{2\sqrt{2}+3}{4}} = \frac{\sqrt{2}+1}{2} \approx 1.2071$

$\frac{1}{2}$

$(1-\lambda)f(\frac{1+\lambda}{2}) + \lambda f(\frac{\lambda}{2})$ $\quad \min = 1.2071 \quad \max = 1.2906.$

$f(\frac{1+\lambda}{2}) = \| \frac{1}{2}\begin{pmatrix} 1+\lambda & 0 \\ 1-\lambda & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1+\lambda & 1-\lambda \\ 0 & 0 \end{pmatrix} \|$ $\quad f(\frac{\lambda}{2}) = \| \frac{1}{2}\begin{pmatrix} \lambda & 0 \\ 2-\lambda & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \lambda & 2-\lambda \\ 0 & 0 \end{pmatrix} \|$

$1-t = \frac{1-\lambda}{2}$

$= \| \begin{pmatrix} 1+\lambda & \frac{1-\lambda}{2} \\ \frac{1-\lambda}{2} & 0 \end{pmatrix} \|$ $\quad = \| \begin{pmatrix} \lambda & \frac{2-\lambda}{2} \\ \frac{2-\lambda}{2} & 0 \end{pmatrix} \|$

$A^*A = \begin{pmatrix} (1+\lambda)^2+\frac{(1-\lambda)^2}{4} & \frac{1-\lambda^2}{2} \\ \frac{1-\lambda^2}{2} & \frac{(1-\lambda)^2}{4} \end{pmatrix}$ $\quad A^*A = \begin{pmatrix} \lambda^2+\frac{(2-\lambda)^2}{4} & \frac{\lambda(2-\lambda)}{2} \\ \frac{\lambda(2-\lambda)}{2} & \frac{(2-\lambda)^2}{4} \end{pmatrix}$

$\sqrt{\lambda_{max}} = \sqrt{\frac{3x^2+2x+3+2\sqrt{2}\sqrt{x^4+2x^3+2x^2+2x+1}}{2}}$

$\sqrt{\lambda_{max}} = \sqrt{\frac{4-4x+3x^2+2\sqrt{2}\sqrt{x^4-2x^3+2x \ldots}}{2}}$

$2t^2-\frac{9}{5}t+\frac{4}{5} \ge 0$

推论

$f(t)$ 凸函数

令 $v=1, S=T^*$

则 $f(t)=\|tT+(1-t)T^*\|$

是凸函数

$f_v(T)=\|$

$\frac{9}{10}$ $\frac{1}{2}$

$2t^2-2t+1 > \frac{81}{100}-\frac{9}{10}t+\frac{1}{4}t^2$

$\frac{7}{4}t^2-\frac{11}{10}t+\frac{19}{100}>0$

$\frac{7}{4}(t^2-\frac{44}{70}t+\frac{121}{1225})+\frac{19}{100}-\frac{}{4900}$

$v[tT+(1-t)S-t_0T-(1-t_0)S]$

$=v[(t-t_0)T+(-t+t_0)S]$

连续性

$\left| \|tT+(1-t)T^*\|-\|t_0T+(1-t_0)T^*\| \right| \leq$

$tT+(1-t)T^* \to t_0T+(1-t_0)T^*$

$\|tT+(1-t)T^*-t_0T-(1-t)T^*\|$

$\leq |t-t_0|\,\|T\|+|1-t-1+t_0|\,\|T^*\|$

$v=\frac{1}{2}$ $t=\frac{1+\lambda}{2}$

$f_v(t)=\|T+S\|_v=\|v[tT+(1-t)S]+(1-v)[tT^*+(1-t)S^*]$

$\frac{1}{2}\left(\begin{pmatrix}1+\lambda & 0\\ 0 & 0\end{pmatrix}+\begin{pmatrix}0 & 0\\ 1-\lambda & 0\end{pmatrix}\right)+\frac{1}{2}\left(\begin{pmatrix}1+\lambda & 0\\ 0 & 0\end{pmatrix}+\begin{pmatrix}1-\lambda\\ 0\end{pmatrix}\right)$

$\begin{pmatrix}1+\lambda\\ 1-\lambda\end{pmatrix}$

$\Delta=(1+\lambda)^4+2(1+\lambda)^2(1-\lambda)^2+\frac{(1-\lambda)^4}{4}-\frac{(1-\lambda)^4}{4}+(1+\lambda)^2(1-\lambda)^2$

$=(1+\lambda)^2\left((1+\lambda)^2+2(1-\lambda)^2\right)$

$(1+\lambda)^2+\frac{(1-\lambda)^2}{2}+(1+\lambda)\sqrt{\lambda^2+2\lambda+1+2\lambda^2-4\lambda+2}$

$\frac{2(1+\lambda)^2+(1-\lambda)^2+2(1+\lambda)\sqrt{3\lambda^2-2\lambda+3}}{4}$

f

$(1-\lambda) f\left(\frac{1+\lambda}{2}\right) + \lambda f\left(\frac{\lambda}{2}\right)$ $T = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ $S = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ $v = \frac{1}{2}$

$\left(\frac{1+\lambda}{2}\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} + \ldots\right) = (1-\lambda)\sqrt{\frac{2(\lambda+1)^2 + (\lambda-1)^2 + 2(\lambda+1)\sqrt{2\lambda^2+2}}{4}}$

$A = \begin{pmatrix} 1+\lambda & \frac{1-\lambda}{2} \\ \frac{1-\lambda}{2} & 0 \end{pmatrix}$ $+\lambda$ $\lambda \left\| \begin{pmatrix} \lambda & \frac{2-\lambda}{2} \\ 2-\lambda & 0 \end{pmatrix} \right\|$

$A^* \begin{pmatrix} 1+\lambda & \frac{1-\lambda}{2} \\ \frac{1-\lambda}{2} & 0 \end{pmatrix}$ a^2 b^2

$A^*A = \begin{pmatrix} (\lambda+1)^2 + \frac{(1-\lambda)^2}{4} & \frac{(1+\lambda)(1-\lambda)}{2} \\ \frac{(1+\lambda)(1-\lambda)}{2} & \frac{(1-\lambda)^2}{4} \end{pmatrix}$ $\begin{pmatrix} \lambda^2 + \frac{(2-\lambda)^2}{4} & \frac{(2-\lambda)\lambda}{2} \\ \frac{(2-\lambda)\lambda}{2} & \frac{(2-\lambda)^2}{4} \end{pmatrix}$

$\left(a^2 + \frac{b^2}{4} - \lambda\right)\left(\frac{b^2}{4} - \lambda\right) - \frac{a^2b^2}{4} = 0$

$\cancel{\frac{a^2b^2}{4}} + \frac{b^4}{16} - \frac{b^2}{4}\lambda - a^2\lambda - \frac{b^2}{4}\lambda + \lambda^2 - \cancel{\frac{a^2b^2}{4}} = 0$ $\begin{pmatrix} a^2 + \frac{b^2}{4} & \frac{ab}{2} \\ \frac{ab}{2} & \frac{b^2}{4} \end{pmatrix}$

$\lambda^2 - \lambda\left(a^2 + \frac{b^2}{2}\right) + \frac{b^4}{16} = 0$ $2\lambda^2 + 4\lambda + 2$ $\lambda^2 - 2\lambda + 1$

$= \frac{(1-\lambda)\sqrt{4(1+\lambda)^2 + (1-\lambda)^2 + 2(1+\lambda)\sqrt{2\lambda^2+2}}}{2}$ $\Delta = a^4 + a^2b^2 + \frac{b^4}{4} - \frac{b^4}{4} = 0$ $= \sqrt{\frac{2(\lambda+1)^2 + (\lambda-1)^2 + 2(\lambda+1)\sqrt{2\lambda^2+2}}{\ldots}}$

$\Delta = a^2(a^2+b^2)$

$+ \frac{\lambda(2-\lambda)\sqrt{2\lambda^2+4+2\lambda\sqrt{\lambda^2+4}}}{4}$ $\frac{a^2 + \frac{b^2}{2} + a\sqrt{a^2+b^2}}{2}$ $= \frac{2a^2 + b^2 + 2a\sqrt{2\lambda^2+2}}{2}$

$(a^2+b^2-\lambda)(b^2-\lambda) - a^2b^2 = 0$

a^2+b^2 ab

ab b^2

$b = \frac{(2-\lambda)}{2}$ $\lambda^2 - (a^2b^2 + 2b^2)\lambda + b^4 = 0$

$\Delta = a^4b^4 + 4a^2b^4 + 4b^4 - 4b^4 = 0$

$\left(a^2 + \frac{b^2}{4} - x\right)\left(\frac{b^2}{4} - x\right) - \frac{a^2b^2}{4}$ $b^2 = \frac{(2-\lambda)^2}{2}$

$\frac{a^2b^2 + 2b^2 + ab^2\sqrt{a^2+4}}{2}$

$(x - a^2 - b^2)(x - b^2) - a^2b^2 = 0$

$x^2 - (2b^2 + a^2)x + b^4$ $-\frac{b^2}{4}x + x^2 = 0$ $\frac{\lambda^2(2-\lambda)^2}{4} + \frac{2(2-\lambda)^2}{4} + \frac{\lambda(2-\lambda)^2}{4}$

$\Delta = a^4 + 4a^2b^2 + 4b^4 - 4b^4$ $x^2 - \left(a^2 + \frac{b^2}{2}\right)x + \frac{b^4}{16} = 0$

$= a^4 + 4a^2b^2$

$\lambda = \frac{a^2 + 2b^2 + a\sqrt{a^2+4b^2}}{2}$ $\frac{b^2}{2}(a^2 + 2 + a\sqrt{a^2+4})b^2$ $\frac{(2-\lambda)^2}{4} \cdot \frac{(\lambda^2 + 2 + \lambda\sqrt{\lambda^2+4})}{2}$

$\frac{\lambda^2 + \frac{(2-\lambda)^2}{2} + \lambda\sqrt{\lambda^2 + (2-\lambda)^2}}{2}$ $\frac{\lambda(2-\lambda)\sqrt{2\lambda^2+4+2\lambda\sqrt{\lambda^2+4}}}{4}$

$\frac{b^2(b^2+1)}{4}$

$= \frac{2\lambda^2 + (2-\lambda)^2 + 2\lambda\sqrt{\lambda^2 + (2-\lambda)^2}}{4}$

$$\int_0^{\frac{1}{2}} \sqrt{2x^2 - \frac{1}{5}x + 1}\, dx$$

$$\sqrt{2x^2 - \frac{1}{5}x + 1} = \frac{t^2 + \frac{1}{5}t + 2}{2 - t^2} \qquad dx = \frac{2t^2 + \frac{2}{5}t + 4}{(2-t)^2}\, dt$$

$$= \int_0^{\frac{1}{2}} \frac{2(t^2 + \frac{1}{5}t + 2)^2}{(2-t^2)^3}\, dt$$

$$\frac{2t^2+2t+2}{(2t+1)^2} dt$$

$$\frac{2t+1}{}$$

$$t[\nu S + (1-\nu)S^*] + (1-t)[\nu T + (1-\nu)T^*]$$

7/18.

Zamani 定理 3.9 (i) $\|F_\nu(T)\| = \|T\|$ ＊数分 确界原理

(ii) $\lim_{n\to\infty} \langle Tx_n, T^* x_n\rangle = \|T\|^2$

$$\|\nu T + (1-\nu)T^*\| = \|F_\nu(T)\| = \sup_{\|x\|=1} \|(\nu T + (1-\nu)T^*)x\| = \|T\|$$

$\exists \{x_n\}$ 成立

$$\lim_{n\to\infty} \|\nu T x_n + (1-\nu)T^* x_n\| = \|T\|$$

$$\|\nu T x_n + (1-\nu)T^* x_n\| \quad (n\to\infty)$$

$$= \nu\|Tx_n\| + (1-\nu)\|T^*\| \quad (n\to\infty)$$

$$= \nu\|T\| + (1-\nu)\|T^*\| = \|T\|$$

$$\Rightarrow \lim_{n\to\infty} \|Tx_n\| = \|T\|$$

类似地 $\lim_{n\to 0} \|T^* x_n\| = \|T\|$

$$\Rightarrow 2\nu(1-\nu)\|T\|^2$$

$$(\mathrm{Re}\langle Tx_n, T^* x_n\rangle)^2 + (\mathrm{Im}\langle Tx_n, T^* x_n\rangle)^2 = 2\nu(1-\nu)\,\mathrm{Re}\langle Tx_n + T^* x_n\rangle$$

$$(n\to\infty)$$

$$= |\langle Tx_n, T^* x_n\rangle|^2$$

$$\le \|T\|^2\|T^*\|^2 = \|T\|^4$$

$$\Rightarrow \lim_{n\to\infty} \mathrm{Im}\langle Tx_n, T^* x_n\rangle = 0$$

$$\Rightarrow \lim_{n\to\infty} \mathrm{Re}\langle Tx_n, T^* x_n\rangle = \|T\|^2$$

$\|T\|^2 = \|vTx_n + (1-v)T^*x_n\|^2$

$= \langle vTx_n + (1-v)T^*x_n, vTx_n + (1-v)T^*x_n \rangle$

$= v^2\|Tx_n\|^2 + \langle vTx_n, (1-v)T^*x_n \rangle + \langle (1-v)T^*x_n, Tx_n \rangle + (1-v)^2\|T^*x_n\|^2$

$= v^2\|Tx_n\|^2 + 2v(1-v)\mathrm{Re}\langle Tx_n, T^*x_n \rangle + (1-v)^2\|T^*x_n\|^2$

$= v^2\|T\|^2 + (1-v)^2\|T\|^2 + 2v(1-v)\mathrm{Re}\langle Tx_n, T^*x_n \rangle \quad (n \to \infty)$

$= (2v^2 + 1 - 2v)\|T\|^2 +$

略缩

(ii) → (i) $\lim\limits_{n\to\infty} \mathrm{Re}\langle Tx_n, T^*x_n \rangle = \|T\|^2$

$|\langle Tx_n, T^*x_n \rangle| \le \|Tx_n\| \|T^*x_n\|$

$\le \|Tx_n\| \|T^*\|$

($n\to\infty$ 时取等) $\le \|T\|^2$

$\lim\limits_{n\to\infty} \|Tx_n\| = \|T\|$

$\lim\limits_{n\to\infty} \|T^*x_n\| = \|T\|$

$\lim\limits_{n\to\infty} a_n + ib_n = a \in \mathbb{R}$

记 $\lim a_n = a$

$\forall \varepsilon\ \exists N, \forall n > N$, s.t

$|a_n + ib_n - a| < \varepsilon$

$(a_n - a)^2 + (b_n)^2 < \varepsilon^2$

$\Rightarrow |a_n - a| < \varepsilon$

$\|T\|^2 = v^2 \lim\limits_{n\to\infty} \|Tx_n\|^2 + 2(v-v)^2 \lim\limits_{n\to\infty} \mathrm{Re}\langle Tx_n, T^*x_n \rangle + (1-v)^2 \lim\limits_{n\to\infty} \|T^*x_n\|^2$

$= \lim\limits_{n\to\infty} \|F_v(T)x_n\|^2 = \|F_v(T)\|^2 = \|T\|^2$

$\Rightarrow \|F_v(T)\| = \|T\|$

Thrm 2.4. $\|T+S\|_v = \|T\|_v + \|S\|_v$ (ii)

Cor. 2.5 (1) $\sim$ (2.4)

(2) $\|S^*T\|_v = \|S\|_v \|T\|_v$

Thrm. 2.7. $\|T+S\|_v = 2\max\{\|T\|_v, \|S\|_v\}$ (ii)

Thm 2.8. ⑴朋 (i) 朋 (ii)

Prop. 3.9. $w_v(T^2) \le w_v^2(T)$

7/24.

写 (i)(ii) 例子　确界原理（数分，华师大）　P.第3,4页

Thm 2.4. $\|T+S\|_v = \|T\|_v + \|S\|_v$

$\|(T+S)x_n\|^2$
$= \|Tx_n\|^2 + 2\mathrm{Re}\langle Tx_n, Sx_n\rangle + \|Sx_n\|^2$

$\|T+S\|^2 \le \|T\|^2 + \|S\|^2 + \lim\limits_{n\to\infty} 2\mathrm{Re}\langle Tx_n, Sx_n\rangle$
$= \|T\|^2 + \|S\|^2 + 2\|T\|\|S\|$

$\Rightarrow \|T\|\|S\| \le \lim\limits_{n\to\infty} \mathrm{Re}\langle Tx_n, Sx_n\rangle$

Proof　(1)

$= \lim\limits_{n\to\infty} \|(T+S)x_n\|$

$= \lim\limits_{n\to\infty} \left(\underbrace{\|Tx_n\|^2}_{a_n} + 2\underbrace{\mathrm{Re}\langle Tx_n, Sx_n\rangle}_{c_n} + \underbrace{\|Sx_n\|^2}_{b_n}\right)$　(1)

$\le \|T\|^2 + \|S\|^2 + 2\|T\|\|S\|$

$\lim\limits_{n\to\infty} a_n$, $\lim\limits_{n\to\infty} b_n$ 极限存在

故 $\lim\limits_{n\to\infty} c_n = \lim\limits_{n\to\infty} (a_n + b_n + c_n) - \lim\limits_{n\to\infty} a_n - \lim\limits_{n\to\infty} b_n$ 极限存在

$\Rightarrow \lim\limits_{n\to\infty} \mathrm{Re}\langle Tx_n, Sx_n\rangle$ 极限存在

又因为(1)存在 $\lim\limits_{n\to\infty} \|(T+S)x_n\|^2 = (\|T\| + \|S\|)^2$
$= \|T\|^2 + \|S\|^2 + 2\|T\|\|S\|$

$\Rightarrow \lim\limits_{n\to\infty} \mathrm{Re}\langle Tx_n, Sx_n\rangle = \|T\|\|S\|$

存在 $\|(v(T+S) + (1-v)(T+S)^*)x_n\| = \|v(T+S) + (1-v)(T+S)^*\|$

$\le$

$\|(v(T+S) + (1-v)(T+S)^*)x_n\| \le \|(vT + (1-v)T^*)x_n\| + \|vS + (1-v)\ldots$

$\|(vT + (1-v)T^*)x_n + (vS + (1-v)S^*)x_n\| \le \|vT + (1-v)T^*\| + \|vS + (1-v)S^*\|$ 极限存在

$\|v(T+S)x_n + (1-v)(T+S)^*x_n\|^2$

$= \|v(T+S)x_n\|^2 + 2\mathrm{Re}\langle v(T+S)x_n, (1-v)(T+S)^*x_n\rangle + \|(1-v)(T+S)^*x_n\|^2$

$\le \|v(T+S)\|^2 + 2\|v(T+S)\|\|(1-v)(T+S)^*\| + \|(1-v)(T+S)^*\|$

所以 $\lim\limits_{n\to\infty} \mathrm{Re}\langle v(T+S)x_n, (1-v)(T+S)^*x_n\rangle$ 极限存在

$\lim \|(v(T+S)+(1-v)(T+S)^*)x_n\| = (\|v(T+S)\| + \|(1-v)(T+S)^*\|)^2$

$= \|v(T+S)\|^2 +$

$\lim_{n\to\infty} 2\mathrm{Re}\langle v(T+S)x_n, (1-v)(T+S)^* x_n\rangle = \|v(T+S)\| \, \|(1-v)(T+S)^*\|$

$\lim_{n\to\infty} \mathrm{Re}\langle F_v(T)x_n, F_v(S)x_n\rangle = \|F_v(T)\| \, \|F_v(S)\|$

$\lambda(2\cos\theta - \lambda) - 1 = 0$

$-\lambda^2 \cos\theta + \lambda^2 - 1 = 0$ $\quad F_v(S)$

$\lambda^2 - 2\lambda\cos\theta + 1$

$\lambda_{max} = \cos\theta + \sqrt{\cos^2\theta + 1}$

$\|F_v(e^{i\theta}T)\|$

$= \|v e^{i\theta} T + (1-v) e^{-i\theta} T^*\|$

$= \left\| \begin{pmatrix} e^{i\theta} + e^{-i\theta} & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix} \right\|$

$A^* = \begin{pmatrix} e^{i\theta} + e^{-i\theta} & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix}$

$\begin{vmatrix} 4 & 4 \\ 4 & 4 \end{vmatrix}$

$T\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix} \quad v = \frac{1}{2} \qquad (T^*T)^{\frac{1}{2}}$

$T^*\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} \quad T^*T\begin{pmatrix} 8 & 0 \\ 0 & 0 \end{pmatrix}$

$|T| = \begin{pmatrix} 2\sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{pmatrix}$

$|T^*| = \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{pmatrix}$

$A^*A = \begin{pmatrix} 4\cos(\theta)^2 + e^{i\theta} & e^{-2i\theta} + 1 \\ e^{-2i\theta} + 1 & e^{2i\theta} \end{pmatrix}$

$\left\| \begin{pmatrix} \frac{3\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \right\| + \left\| \begin{pmatrix} \frac{3\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \right\|$

$(4\cos(\theta)e^{i} + e^{-4i\theta} - x)(e^{-2i\theta} - x) - 4\cos(\theta)^2 e^{-2i\theta} = 0$

$4\cos(\theta)e^{-i\theta} + e^{-4i\theta} - (4\cos(\theta)e^{i\theta} + 2e^{-2i\theta})x + x^2 - 4 = 0$

$A^*A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$

$-\lambda(8-\lambda) = 0$

$\lambda = 8 \quad \sqrt{\lambda} = 2\sqrt{2}$

$(5-\lambda)(1-\lambda) - 4 = 0$

$(4\cos(\theta)e^{i\theta} + e^{2i\theta} - \lambda)(e^{2i\theta} - \lambda) - 4(\cos(\theta))^2 e^{2i\theta} = 0$

$\lambda^2 - 6\lambda + 1 = 0$

$\lambda^2 - (4\cos(\theta)e^{i\theta} + 2e^{2i\theta})\lambda + 4\cos(\theta)e^{i\theta} + e^{4i\theta}$

$\frac{6 + 4\sqrt{2}}{2}$

$e^{i\theta} + e^{-i\theta} = 2\mathrm{Re}\, e^{i\theta}$

$-4(\cos(\theta))^2 e^{2i\theta} = 0$

$\begin{pmatrix} 2e^{2i\theta} + e^{-2i\theta} + 2 & e^{-2i\theta} + 1 \\ e^{-2i\theta} + 1 & e^{-2i\theta} \end{pmatrix}$

$\lambda_{max} = 3 + 2\sqrt{2}$

$\|\;\| = \sqrt{2} + 1$

$(4\mathrm{Re}(e^{i\theta}) + e^{2i\theta} - \lambda)(e^{-2i\theta} - \lambda) - e^{-4i\theta} - 2e^{-2i\theta} - 1 = 0$

$4\mathrm{Re}(e^{i\theta}) \cdot e^{-2i\theta}$

$(2e^{2i\theta} + e^{-2i\theta} + 2 - \lambda)(e^{-2i\theta} - \lambda) - e^{-4i\theta} - 2e^{-2i\theta} - 1 = 0$

$2 + e^{-4i\theta} + 2e^{-2i\theta} - \lambda$

$\cos(\theta)^2$

2

4

$2 - \lambda e^{-2i\theta} - 2\lambda e^{2i\theta} - \lambda e^{-2i\theta} - 2\lambda + \lambda^2 = 0$

$\lambda^2 - (2e^{-2i\theta} + 2e^{2i\theta} + 2)\lambda + 2 = 0$

$\lambda = 2\cos(2\theta) + 1 + \sqrt{2\cos(4\theta) + 4\cos(2\theta) + 1}$

$\|T+S\| = \|T\| + \|S\|$
$\|S^*T\| = \|S\|\|T\|$

$\geq \omega_v(S^*T) =$

$2t+1+\sqrt{4t^2+4t-1}$

$\|F_v(T)\|\,\|F_v(S)\| \leq \sup \|ve^{i\theta}S^*T + (1-v)e^{-i\theta}ST^*$

$\frac{8t+4}{\sqrt{4t^2+4t-1}} + 4 = 0$

7/29. $w_{\frac{1}{2}}(AB)$

$= w\left(\frac{e^{i\theta}AB + e^{-i\theta}(AB)^*}{2}\right) \leq \omega_v(AB)$

$\frac{2t+1}{\sqrt{4t^2+4t-1}} = -1$

$\frac{4t^2+4t+1}{4t^2+4t-1} = 1$

$w_v(T) = \sup_{\theta\in\mathbb{R}} \|\mathrm{Re}_v(e^{i\theta}T)\|$

$= \sup_{\theta\in\mathbb{R}} \|ve^{i\theta}T + (1-v)e^{-i\theta}T^*\|$

$3+\sqrt{7}$ (第1) (第3)

Thm. 3.4, Prop 3.9 $\omega(T) = \sup_{\theta\in\mathbb{R}} \left\|\frac{e^{i\theta}T + e^{-i\theta}T^*}{2}\right\|$

// 看 $w_v(T^2)$ 与 $w_v^2(T)$ 的关系

$v \in [\frac{1}{2}, 1]$

$w_{\frac{1}{2}}(T) < \omega_v(T) < w_v$

① $\underbrace{\frac{1}{2v} w_v(T^2) \leq w_{\frac{1}{2}}(T^2)}_{\text{Thm. 3.4}} \underbrace{\leq w_{\frac{1}{2}}^2(T)}_{\text{Prop 3.9}} \underbrace{\leq w_v^2(T)}_{\text{极值}}$ 按自己理解写证明

先用 Thm. 3.4 结果

② 调 3.12 (可能)先用 3.11 的结果

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$f_T(v) = w_v(T)$

T和S都是二阶阵 对角阵

$f_T(v) - f_S(v)$
$= \omega_v(T) - w_v(S)$ 什么时候是凸函数

④ Thm 2.8

$g\left(\frac{x+y}{2}\right) \leq \frac{1}{2}g(x) + \frac{1}{2}g(y)$

$= \frac{1}{2}f_T(x) - \frac{1}{2}f_S(x)$
$+ \frac{1}{2}f_T(y) - \frac{1}{2}f_S(y)$

$$\frac{1}{2v} w_v^2(T) = \frac{1}{2v} \| (R(e^{i\theta}T) + i(2v-1) I(e^{i\theta}T)) x \|^2$$

$$w_v(T+S) = w_v(T) + w_v(S)$$

$\lim_{n \to \infty}$

$$w_v(T+S) = w(T) + w(S)$$

$$\sup_{\theta \in R} \left\| \frac{e^{i\theta}(T+S) + e^{-i\theta}(T+S)^*}{2} \right\| = \sup_{\theta \in R}$$

$$\sup_{\theta \in R} \| v e^{i\theta} T + (1-v) e^{-i\theta} T^* + v e^{i\theta} S + (1-v) e^{-i\theta} S^* \|$$

$$\leq$$

$$\leq$$

$$w_v(T+S) = w_v(T) + w_v(S)$$

$$T^* x_n, x_n$$

$\lim$

$$(w_v(T) + w_v(S))^2 = w_v^2(T) + 2 \| v e^{i\theta} T + (1-v) e^{-i\theta} T^* \| \| v e^{i\theta} S + (1-v) e^{-i\theta} S$$

$w_v(T) - w_v(S)$ $\quad T = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ $S = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$

$\| v e^{i\theta} T + (1-v) e^{-i\theta} T^* \|$

$\left\| \begin{pmatrix} ve^{i\theta}a + (1-v)e^{-i\theta}a & 0 \\ 0 & ve^{i\theta}b + (1-v)e^{-i\theta}b \end{pmatrix} \right.$

$\begin{pmatrix} ve^{-i\theta}a + (1-v)e^{i\theta}a & 0 \\ 0 & ve^{-i\theta}b + (1-v)e^{i\theta}b \end{pmatrix}$

$\begin{pmatrix} v(\sin\theta + i\cos\theta)a + (1-v)(\sin\theta - i\cos\theta)a & 0 \\ 2(v(\sin\theta + i\cos\theta) + (1-v)(\sin\theta - i\cos\theta)) & v(\sin\theta + i\cos\theta)b + (1-v)(\sin\theta - i\cos\theta)b \end{pmatrix}$

$\begin{pmatrix} v(\sin\theta - i\cos\theta)a + (1-v)(\sin\theta + i\cos\theta)a & 2(v(\sin\theta - i\cos\theta) + (1-v)(\sin\theta \\ & v(\sin\theta - i\cos\theta)b + (1-v)(\sin\theta + i\cos\theta)b \end{pmatrix}$

$\begin{pmatrix} a^2((1-v)^2 + v^2 + 2v(1-v)(\sin\theta^2 - \cos\theta^2)) & \\ & b^2(v^2 + (1-v)^2 + 2v(1-v)(\sin\theta^2 - \cos\theta^2)) \end{pmatrix}$

$(ve^{i\theta} + (1-v)e^{-i\theta})$

① $A =$

$\sup_{\theta \in \mathbb{R}} |a| \sqrt{(1-v)^2 + v^2 + 2v(1-v)(\sin\theta^2 - \cos\theta^2)}$

条件:

常値

$f(\theta) = \sqrt{\max\{a,b\}^2 - \max\{c,d\}^2}$

$(v(\sin\theta + i\cos\theta) + (1-v)(\sin\theta - i\cos\theta))^2$

② Attempt $T = \begin{pmatrix} a & \\ 2 & b \end{pmatrix}$ $\quad v(\sin\theta - i\cos\theta)a \quad v(\sin\theta + i\cos\theta)a$

$A^*A = \begin{pmatrix} (a^2+4)A - \lambda & 2bA \\ 2bA & b^2A - \lambda \end{pmatrix}$ $\quad 0$ $\quad \begin{pmatrix} a(ve^{i\theta} + (1-v)e^{-i\theta} \\ 2(ve^{i\theta} + (1-v)e^{-i\theta} & b(ve^{i\theta} + (1-v)e^{-i\theta} \end{pmatrix}$

$(\sin\theta^2 - (2v-1)^2\cos\theta^2 + 2i(2v-1)\sin\theta\cos\theta)$

$a^2A + 4A$

$((a^2+4)A - \lambda)(b^2A - \lambda) - 4b^2A^2 = 0$

$\lambda^2 - (a^2+b^2+4)A\lambda + (a^2b^2)A^2 = 0$

$\Delta = (a^2+b^2+4)^2A^2 - 4a^2b^2A^2$

$\Delta = A(a^4 + b^4 + 16 + 2a^2b^2 + 8a^2 + 8b^2)$

$\dfrac{a^2+b^2+4 + \sqrt{\Delta}}{2}$

③

$$\begin{pmatrix} a(ve^{-i\theta}+(1-v)e^{i\theta}) & 2ve^{-i\theta} \\ 2(1-v)e^{i\theta} & b(ve^{-i\theta}+(1-v)e^{i\theta}) \end{pmatrix}\begin{pmatrix} a(ve^{i\theta}+(1-v)e^{-i\theta}) & 2(1-v)e^{-i\theta} \\ 2ve^{i\theta} & b(ve^{i\theta}+(1-v)e^{-i\theta}) \end{pmatrix}$$

$e \quad \sin 2\theta$

$$\begin{pmatrix} \underbrace{a^2(v^2+(1-v)^2+v(1-v)(e^{2i\theta}+e^{-2i\theta}))}_{a^2A}+4v^2 & 2a(v(1-v)e^{-2i\theta}+(1-v)^2)+2b(v^2+v(1-v)e^{-2i\theta}) \\ 2a(v(1-v)e^{2i\theta}+(1-v)^2)+2b(v^2+v(1-v)e^{2i\theta}) & 4(1-v)^2+\underbrace{b^2(v^2+(1-v)^2+v(1-v)(e^{2i\theta}+e^{-2i\theta}))}_{b^2A} \end{pmatrix}$$

$$\lambda^2-(4v^2+4(1-v)^2+a^2A+b^2A)\lambda+$$
$$16v^2(1-v)^2+4v^2b^2A+4(1-v)^2a^2A+a^2b^2A^2-\triangle$$

$$(4a^2(v^2(1-v)^2+(1-v)^4+v(1-v)^3e^{2i\theta}+v(1-v)^3e^{-2i\theta})$$
$$+4ab(v^3(1-v)e^{2i\theta}+2v^2(1-v)^2+v(1-v)^3e^{-2i\theta}+v^3(1-v)e^{-2i\theta}+2v^2(1-v)^2+v(1-v)^3e^{2i\theta})$$
$$+4b^2(v^4+v^2(1-v)^2+v^3(1-v)e^{2i\theta}+v^3(1-v)e^{-2i\theta})$$

$$4a^2((1-v)^4+v^2(1-v)^2+v(1-v)^3\sin 2\theta)$$
$$+4ab(v^3(1-v)\sin 2\theta+v(1-v)^3\sin 2\theta+4v^2(1-v)^2)$$
$$+4b^2(v^4+v^2(1-v)^2+v^3(1-v)\sin 2\theta)$$

$$\Delta = 16v^4+16(1-v)^4+a^4A^2+b^4A^2-32v^2(1-v)^2+8a^2v^2A-8b^2v^2A-8a^2(1-v)^2A$$
$$+8b^2(1-v)^2A-2a^2b^2A^2$$
$$+16a^2(1-v)^4+16a^2v^2(1-v)^2$$
$$+16a^2v(1-v)^3\sin 2\theta+16ab\,v^3(1-v)\sin 2\theta+16ab\,v(1-v)^3\sin 2\theta+64ab\,v^2(1-v)^2$$
$$+16b^2v^4+16b^2v^2(1-v)^2+16b^2v^3(1-v)\sin 2\theta$$

$$\lambda = \frac{4v^2+4(1-v)^2+a^2A+b^2A+\sqrt{\Delta}}{2}$$

$T=\begin{bmatrix}0&a\\b&0\end{bmatrix}$ $\quad S=\begin{bmatrix}0&c\\d&0\end{bmatrix}$

$W_v(T)-W_v(S)$

$\sup_{\theta}\ \ ve^{i\theta}T+ve^{-i\theta}T^*$

$A=\begin{pmatrix}0 & ave^{i\theta}+b(1-v)e^{-i\theta}\\ bve^{i\theta}+a(1-v)e^{-i\theta} & 0\end{pmatrix}$

$A^*=\begin{pmatrix}0 & bve^{-i\theta}+a(1-v)e^{i\theta}\\ ave^{-i\theta}+b(1-v)e^{i\theta} & 0\end{pmatrix}$

$A^*A=\begin{pmatrix}b^2v^2+a^2(1-v)^2+2abv(1-v)\sin 2\theta & 0\\ 0 & a^2v^2+b^2(1-v)^2+abv(1-v)\sin 2\theta\end{pmatrix}$

$\sin 2\theta=1$時 最大. $\begin{pmatrix}(a(1-v)+bv)^2 & 0\\ 0 & (av+b(1-v))^2\end{pmatrix}$

$f_1(v)=a(1-v)+bv-c(1-v)-dv=v(-a+b+c-d)+a-c$ $\quad 0<v<\frac{1}{2},\ a>b,\ c>d$ $\quad \frac{1}{2}<v<1,\ a<b,\ c<d$

$f_2(v)=a(1-v)+bv-cv-d(1-v)=v(-a+b-c+d)+a-d$ $\quad 0<v<\frac{1}{2}:\ a>b,\ c<d$ $\quad \frac{1}{2}<v<1,\ a<b,\ c>d$

$f_3(v)=av+b(1-v)-c(1-v)-dv=v(a-b+c-d)+b-c$ $\quad 0<v<\frac{1}{2}\ \ a<b,\ c>d$ $\quad \frac{1}{2}<v<1\ \ a>b,\ c<d$

$f_4(v)=av+b(1-v)-cv-d(1-v)=v(a-b-c+d)+b-d$ $\quad 0<v<\frac{1}{2}\ \ a<b,\ c<d$ $\quad a>b,\ c>d$

$(b-a)v+a>0$

正 $v>\frac{-a}{a-b}$

負 $v<\frac{-a}{a-b}$

$(a-b)v+b$

正 $v>\frac{-b}{a-b}$

$v<\frac{-b}{a-b}$

$|bv+a(1-v)|>|av+b(1-v)|$

$|(b-a)v+a|>|(a-b)v+b|$

① $(b-a)v+a>(a-b)v+b$

$a>b$時 $0\leq v\leq\frac{1}{2}$

$a<b$時 $\frac{1}{2}\leq v\leq 1$

$0>\frac{-a}{a-b}\quad a<0$

$0>\frac{-b}{b-a}\quad b<0$

② $(b-a)v+a>(b-a)v-b$

$a>-b$

③ $(a-b)v-a>(a-b)v+b$

$a<-b$

④ $(a-b)v-a>(b-a)v-b$

$2(a-b)v>a-b$

$a-b<0$

$a<b$時 $0\leq v\leq\frac{1}{2}$

$a>b$時 $\frac{1}{2}\leq v\leq 1$

$a>b\ \ \frac{1}{2}\leq b\leq 1$

$a<b\ \ \frac{1}{2}\leq v\leq$

$$W_v(T) \le \frac{1}{2}\left(\|T-T^*\|_v^2 + \|T+T^*\|_v^2\right)^{\frac{1}{2}}$$

$$w_v^2(T) \le \frac{1}{4}\left(\|ve^{i\theta}(T-T^*)+(1-v)e^{-i\theta}(T-T^*)\|^2 + \|ve^{i\theta}(T+T^*)+(1-v)e^{-i\theta}(T-T^*)\|\right.$$

$$\langle\|ve^{i\theta}(T-T)^*y\|^2 + 2\mathrm{Re}\langle$$

8.14. ① 原文的2.4作为$v=1$情况的一个推论

改成

② 存在α使得

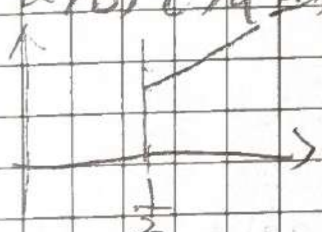
$A=A^*$时

$$\|A\| = \|A\|_v = \|vA+(1-v)A^*\|$$
$$= \|vA^*+(1-v)A^*\|$$
$$= \|A^*\|$$
$$= \|A\|$$

$$\|F_v(T)\| = \|T\|$$

P36. 原则

$a>b,\ c>d$时



$|a-b|>|c-d|$时为凸函数

$|a-b|<|c-d|$时为凹

$a>b,\ c<d$时

$|a-b|>|c-d|$时 为凸函数

$|a-b|<|c-d|$时 为凹函数

$a<b,\ c>d$时

$|a-b|>|c-d|$时为凸

$|a-b|<|c-d|$ 为凹

$a<b,\ c<d$时

$|a-b|>|c-d|$时为凸

$|a-b|<|c-d|$时为凹

$$A=\begin{pmatrix} 0 & ave^{i\theta}+b(1-v)e^{-i\theta} \\ bve^{i\theta}+a(1-v)e^{-i\theta} & 0 \end{pmatrix}$$

$Re^2 a + Im^2 a$

$$A^*=\begin{pmatrix} 0 & \bar{b}ve^{-i\theta}+\bar{a}(1-v)e^{i\theta} \\ \bar{a}ve^{-i\theta}+\bar{b}(1-v)e^{i\theta} & 0 \end{pmatrix}$$

$$\begin{pmatrix} (Re^2 b+Im^2 b)v^2+a\bar{b}v(1-v)e^{2i\theta}+\bar{a}bv(1-v)e^{-2i\theta}+(Re^2a+Im^2a)(1-v)^2 & 0 \\ 0 & (Re^2a+Im^2a)v^2+\bar{a}bv(1-v)e^{-2i\theta}+a\bar{b}v(1-v)e^{2i\theta}+(Re^2b+Im^2b)(1-v)^2 \end{pmatrix}$$

$a\bar{b}+\bar{a}b$

$(Re\,a+iIm\,a)(Re\,b-iIm\,b)$

$(Re\,a\,Re\,b+iRe\,b\,Im\,a-iRe\,a\,Im\,b+Im\,a\,Im\,b)(\sin 2\theta-i\cos 2\theta)$

$(Re\,a\,Re\,b-iRe\,b\,Im\,a+iRe\,a\,Im\,b+Im\,a\,Im\,b)(\sin 2\theta+i\cos 2\theta)$

$(Re\,a-iIm\,a)(Re\,b+iIm\,b)$

$Re\,a\,Re\,b-iRe\,b\,Im\,a+iRe\,a\,Im\,b+Im\,a\,Im\,b$

$(Re^2b+Im^2b)v^2+2(Re\,a\,Re\,b+Im\,a\,Im\,b)+(Re^2a+Im^2a)(1-v)^2$

$2(Re\,a\,Re\,b+Im\,a\,Im\,b)\sin 2\theta\,(v(1-v))$

$$\begin{pmatrix} (Re(b)v+Re(a)(1-v))^2+(Im(b)v+Im(a)(1-v))^2 & 0 \\ 0 & (Re(a)v+Re(b)(1-v))^2+(Im(a)v+Im(b)(1-v))^2 \end{pmatrix}$$

$|Re(a)v+Re(b)|$

$f(v)=\sqrt{(Re(b)v+Re(a)(1-v))^2+(Im(b)v+Im(a)(1-v))^2}-\sqrt{(Re(a)v+Re(b)(1-v))^2+(Im(a)v+Im(b)(1-v))^2}$

$Re^2(a)v^2+2Re(a)Re(b)(v-v^2)+(v^2-2v+1)Re^2(b)$

$+Im^2(a)v^2+2Im(a)Im(b)(v-v^2)$

$+(v^2-2v+1)Re^2(b)$

$[(Re(a)-Re(b))^2+(Im(a)-Im(b))^2]v^2$

$+2v(Re(a)Re(b)-Re(b)^2+Im(a)Im(b)-Im^2(b))$

$p(z)=z^5+2z^4-2$

$$C_p-\lambda=\begin{pmatrix} -1-\lambda & -1 & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0-\lambda \end{pmatrix}$$

$$C_p'=\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(-1-\lambda)\begin{pmatrix} -\lambda & 0 & 0 & 0 & 0 \\ 1 & -\lambda & 0 & 0 & 0 \\ 0 & 1 & -\lambda & 0 & 0 \\ 0 & 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 1 & -\lambda \end{pmatrix}$$

$$C^*(p)C_p=\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & -2 \\ 1 & 2 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -2 & -2 & 0 & 0 & 0 & 4 \end{pmatrix}$$

$$(-1-\lambda)(-\lambda)\begin{pmatrix} -\lambda & 0 & 0 & 0 \\ 1 & -\lambda & 0 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 1 & -\lambda \end{pmatrix}$$

$(-1-\lambda)(-\lambda)$

C

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 & 4 \end{pmatrix}$$

矩阵的正交对角化

$u(\ddots)u^*$

高等代数 P406

二次型的化简