Abstract

References [1] and [2] have recently defined the weighted numerical radius of operators in different ways. This paper studies the differences and unifications of these two definitions, as well as their connection to the classical numerical radius. Regarding the numerical radius in reference [2],

$$\omega_v(A) = \sup_{\theta \in \mathbb{R}} ||ve^{i\theta}A + (1-v)e^{-i\theta}A^*||$$

where $v \in [0,1]$ and A is any bounded linear operator on a Hilbert space $\mathcal{H}$. Inspired by the definition of weighted operator norms in [1], and based on [2], we provide a new definition of a weighted operator norm. We call

$$M_v(A) = vA + (1-v)A^*$$

the **weighted average transformation** of operator A, and call

$$||A||_v \triangleq ||M_v(A)||$$

the **weighted operator norm** of operator A. We utilize the convexity of the weighted numerical radius combined with the Hadamard inequality to develop estimates for the numerical radius, particularly focusing on inequalities for the weighted numerical radius. We establish necessary and sufficient conditions for some boundary equalities such as

$$||A+B||_v{}^2 = ||A||_v^2 + ||B||_v^2$$

and

$$\omega_v(A^2+B^2) = 4\alpha \max\{\omega_v^2(A), \omega_v^2(B)\}$$

In particular, we use Example 3 to illustrate that the inequality $\omega_v(A+B) \leq \omega_v(A) + \omega_v(B)$ strengthens the results of Carmichael and Mason regarding polynomial root estimation.

Keywords: Weighted numerical radius, Weighted average transformation, Weighted operator norm, Polynomial roots.